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Different representations for the action principle in 4D $\mathcal{N} = 2$ supergravity

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ABSTRACT: Within the superspace formulation for four-dimensional $\mathcal{N} = 2$ matter-coupled supergravity developed in arXiv:0805.4683, we elaborate two approaches to reduce the superfield action to components. One of them is based on the principle of projective invariance which is a purely $\mathcal{N} = 2$ concept having no analogue in simple supergravity. In this approach, the component reduction of the action is performed *without* imposing any Wess-Zumino gauge condition, that is by keeping intact all the gauge symmetries of the superfield action, including the super-Weyl invariance. As a simple application, the c-map is derived for the first time from superfield supergravity. Our second approach to component reduction is based on the method of normal coordinates around a submanifold in a curved superspace, which we develop in detail. We derive differential equations which are obeyed by the vielbein and the connection in normal coordinates, and which can be used to reconstruct these objects, in principle in closed form. A separate equation is found for the super-determinant of the vielbein $E = \text{Ber}(E_M^A)$, which allows one to reconstruct E without a detailed knowledge of the vielbein. This approach is applicable to any supergravity theory in any number of space-time dimensions. As a simple application of this construction, we reduce an integral over the curved $\mathcal{N} = 2$ superspace to that over the chiral subspace of the full superspace. We also give a new representation for the curved projective-superspace action principle as a chiral integral.

KEYWORDS: Extended Supersymmetry, Superspaces, Supergravity Models

ARXIV EPRINT: [0812.3464](https://arxiv.org/abs/0812.3464)

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1 Introduction

One of the main virtues of superspace approaches to supergravity theories in diverse dimensions is the possibility to write down the most general locally supersymmetric actions formulated in terms of a few superfield dynamical variables possessing, as a rule, a transparent geometric origin. The price to pay for this generality is that working out a reduction from the parental superfield action to its component counterpart requires some special care. Being trivial conceptually, such a reduction may be technically quite involved and challenging.

The present paper is aimed at carrying out a component reduction, as well as a partial superspace reduction, for the action principle occurring within the superspace formulation for four-dimensional $\mathcal{N} = 2$ matter-coupled supergravity recently developed in [1], as a

natural extension of the earlier construction for 5D $\mathcal{N} = 1$ supergravity [2, 3]. The matter fields in [1] are described in terms of covariant projective multiplets which are curved-space versions of the superconformal projective multiplets [4] living in rigid projective superspace [5]. In addition to the local $\mathcal{N} = 2$ superspace coordinates¹ $z^M = (x^m, \theta_i^\mu, \bar{\theta}_{\dot{\mu}}^i)$, such a supermultiplet, $Q^{(n)}(z, u^+)$, depends on auxiliary isotwistor variables $u_i^+ \in \mathbb{C}^2 \setminus \{0\}$, with respect to which $Q^{(n)}$ is holomorphic and homogeneous, $Q^{(n)}(c u^+) = c^n Q^{(n)}(u^+)$, on an open domain of $\mathbb{C}^2 \setminus \{0\}$ (the integer parameter n is called the weight of $Q^{(n)}$). In other words, such superfields are intrinsically defined in $\mathbb{C}P^1$. The covariant projective supermultiplets are required to be annihilated by half of the supercharges,

$$\mathcal{D}_\alpha^+ Q^{(n)} = \bar{\mathcal{D}}_{\dot{\alpha}}^+ Q^{(n)} = 0, \quad \mathcal{D}_\alpha^+ := u_i^+ \mathcal{D}_\alpha^i, \quad \bar{\mathcal{D}}_{\dot{\alpha}}^+ := u_i^+ \bar{\mathcal{D}}_{\dot{\alpha}}^i, \quad (1.1)$$

with $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha^i, \bar{\mathcal{D}}_{\dot{\alpha}}^i)$ the covariant superspace derivatives. The dynamics of supergravity-matter systems are described by locally supersymmetric actions of the form [1]:

$$S = \frac{1}{2\pi} \oint_C (u^+ du^+) \int d^4x d^4\theta d^4\bar{\theta} E \frac{W\bar{W}\mathcal{L}^{++}}{(\Sigma^{++})^2}, \quad E^{-1} = \text{Ber}(E_A^M), \quad (1.2)$$

where

$$\Sigma^{++} := \frac{1}{4} \left((\mathcal{D}^+)^2 + 4S^{++} \right) W = \frac{1}{4} \left((\bar{\mathcal{D}}^+)^2 + 4\bar{S}^{++} \right) \bar{W} = \Sigma^{ij} u_i^+ u_j^+. \quad (1.3)$$

Here the Lagrangian $\mathcal{L}^{++}(z, u^+)$ is a covariant real projective multiplet of weight two, $W(z)$ is the covariantly chiral field strength of an Abelian vector multiplet, $S^{++}(z, u^+) = S^{ij}(z) u_i^+ u_j^+$ and $\bar{S}^{++}(z, u^+) = \bar{S}^{ij}(z) u_i^+ u_j^+$ are special dimension-1 components of the torsion. The action (1.2) can be shown to be invariant under the supergravity gauge transformations, and it is also manifestly super-Weyl invariant [1]. It can also be rewritten in the equivalent form

$$S = \frac{1}{2\pi} \oint_C (u^+ du^+) \int d^4x d^4\theta d^4\bar{\theta} E \frac{\mathcal{L}^{++}}{S^{++}\bar{S}^{++}} \quad (1.4)$$

in which, however, the super-Weyl invariance is not manifest. The latter form makes transparent the fact that the action is independent of the compensating vector multiplet described by W and \bar{W} provided \mathcal{L}^{++} is independent of it.

As argued in [1, 6], the dynamics of a general $\mathcal{N} = 2$ supergravity-matter system can be described by an action of the form (1.2), including the chiral actions which can always be brought to the form (1.2). This is why the action principle (1.2) is of fundamental importance in $\mathcal{N} = 2$ supergravity.

There are two special properties of the action (1.2) that we would like to point out. First of all, the integration in (1.2) is carried out over the *full* superspace, therefore one has to integrate out eight Grassmann variables in order to reduce the action to components. Secondly, the Lagrangian in (1.2) obeys the analyticity constraints (1.1) which enforce \mathcal{L}^{++} to depend on only *half* of the superspace Grassmann variables. In this respect, the $\mathcal{N} = 2$

¹World indices take values $m = 0, 1, \dots, 3$, $\mu = 1, 2$, $\dot{\mu} = 1, 2$ and $i = \underline{1}, \underline{2}$, and similarly for tangent space indices; see appendix A for our notation and conventions.

action (1.2), or more precisely its equivalent form (1.4), is analogous to the chiral action in 4D $\mathcal{N} = 1$ supergravity [7, 8], as specially emphasised in [9]. These two features of the $\mathcal{N} = 2$ supergravity action hint at an opportunity to use the experience gained and the techniques developed, e.g., in 4D $\mathcal{N} = 1$ superfield supergravity, in order to reduce (1.2) to components.

In textbooks on 4D $\mathcal{N} = 1$ supergravity [10–12], one can find two methods of component reduction. One of them (to be referred to as *method 1*), elaborated in detail² in [10, 11], was originally introduced by Wess and Zumino [13] and presents itself as a version of the Noether procedure. It involves the following two steps:

- (i) starting from the superfield dynamical variables, one first reads off corresponding multiplets of component fields and their local supersymmetry transformations, using a Wess-Zumino gauge imposed on the superfield vielbein and connection;
- (ii) after that, the desired density multiplet is iteratively reconstructed from its lowest component in conjunction with the known supersymmetry transformation laws.

This method was further developed, and generalized to the case of chiral actions in $\mathcal{N} = 2$ supergravity, in [16–18] using covariant expansions with respect to Θ -variables [10, 13] of somewhat mysterious geometric origin. The other approach (*method 2*) was elaborated in detail in [12], although its first application in the case of pure supergravity was given by Gates and Siegel [8]. It can be implemented provided there exists a formulation of the given supergravity theory in terms of unconstrained prepotentials, and such a formulation is indeed available in the case of 4D $\mathcal{N} = 1$ supergravity [8, 19]. It involves the same step (i) as above modulo the fact that a Wess-Zumino gauge is now imposed on the supergravity prepotentials. Its real gain is that, instead of carrying out the painfully laborious procedure (ii) of method 1, now one should simply do an ordinary Grassmann integral.

Both methods discussed above are hardly of any practical use in the case of $\mathcal{N} = 2$ supergravity formulation under consideration. Being applicable in principle, method 1 becomes too laborious to be used for general $\mathcal{N} = 2$ supergravity-matter systems. As to method 2, no prepotential formulation is yet available for the projective-superspace formulation for $\mathcal{N} = 2$ supergravity given in [1]. A prepotential formulation for $\mathcal{N} = 2$ supergravity has been constructed within the harmonic-superspace approach [20–22].³ However, no comprehensive analysis of the component reduction in curved harmonic superspace has yet appeared.

A relatively new paradigm for component reduction in supergravity appeared some ten years ago. As advocated in refs. [15, 25], which built on the earlier work [26], an ideal means to perform covariant theta-expansions and integrate out Grassmann variables is provided by the superspace normal coordinates introduced a quarter of a century ago by McArthur [27]

²More precisely, ref. [11] only stated the density formula and sketched its derivation. Years later, three of the authors of [11] came up with simple alternative derivations of the density formula [14, 15].

³In the rigid supersymmetric case, the harmonic [20] and the projective [5, 23] approaches are closely related [24], and this should extend, in principle, to the case of supergravity.

for completely different aims.⁴ This technique was applied in [15, 25] to compute the density formula for several supergravity models in diverse dimensions including the case of 4D $\mathcal{N} = 1$ supergravity. Since the method of fermionic normal coordinates employed in [15, 25] is a version of Wess-Zumino gauge in curved superspace, this construction is ultimately related to the earlier approaches pursued in [16–18].

The powerful property of the method of normal coordinates⁵ [27] is its universality, as emphasized in [15] (of course, this is not accidental, for the method is a superspace extension of the Riemann normal coordinates). It can be used for any supergravity theory formulated in superspace, for any number of space-time dimensions. For example, it has recently been used in the case of eleven dimensional supergravity [31]. In particular, it can be applied to reduce the action (1.2) to components. However, the latter application would still require a nontrivial computational effort. Remarkably, the specific feature of 4D $\mathcal{N} = 2$ supergravity (and also 5D $\mathcal{N} = 1$ supergravity) is that it offers us an alternative and much more efficient scheme to reduce the action (1.2) to components which is based on the principle of projective invariance [2, 32, 33]. This unusual invariance, which has no analogue in the $\mathcal{N} = 1$ case, is easy to visualize in a flat superspace limit where the action (1.2) reduces to

$$\begin{aligned}
 S_{\text{flat}} &= \frac{8}{\pi} \oint (u^+ du^+) \int d^4x d^4\theta d^4\bar{\theta} \frac{W\bar{W}L^{++}(u^+)}{(D^+)^2 W (\bar{D}^+)^2 \bar{W}} \\
 &= \frac{1}{2\pi} \oint \frac{(u^+ du^+)}{(u^+ u^-)^4} \int d^4x (D^-)^2 (\bar{D}^-)^2 L^{++}(u^+) \Big|_{\theta=\bar{\theta}=0} .
 \end{aligned}
 \tag{1.5}$$

Here the spinor derivatives D_{α}^- and $\bar{D}_{\dot{\alpha}}^-$ are obtained from D_{α}^+ and $\bar{D}_{\dot{\alpha}}^+$ by replacing $u_i^+ \rightarrow u_i^-$, with the latter being a fixed constant isotwistor for which the only constraint is $(u^+ u^-) \neq 0$ at each point of the integration contour. Since L^{++} is a weight-two rigid projective supermultiplet, the action can be seen to be invariant under arbitrary *projective transformations* of the form:

$$(u_i^-, u_i^+) \rightarrow (u_i^-, u_i^+) R, \quad R = \begin{pmatrix} a & 0 \\ b & c \end{pmatrix} \in \text{GL}(2, \mathbb{C}) .
 \tag{1.6}$$

Clearly, this projective invariance is almost obvious in flat superspace. In curved superspace, however, it turns into a powerful constructive principle to reduce the action (1.2) to components, and what is most non-trivial – without imposing any Wess-Zumino gauge condition!

⁴In [28], the normal coordinate techniques [27] were applied to compute the so-called b_4 (or, equivalently, a_2) coefficients for chiral matter in 4D $\mathcal{N} = 1$ supergravity. Although there exists a purely covariant and very efficient approach to evaluate the Schwinger-DeWitt coefficients in curved superspace [29], the method of superspace normal coordinates [27] proves to be truly indispensable for deriving the density formulae in supergravity theories, as emphasized in [15].

⁵In $\mathcal{N} = 1$ supergravity, there exists a different normal coordinate construction [30] based on the prepotential formulation due to Ogievetesky and Sokatchev [19]. This normal gauge should possess a natural extension to the case of $\mathcal{N} = 2$ supergravity formulated in harmonic superspace [20–22], and it would be very interesting to work out such an extension explicitly.

This paper is organized as follows. In section 2, we provide an alternative derivation of normal coordinates around a submanifold in an arbitrary curved superspace. Although the consideration given in [15] involves some ingenious acrobatics, it leaves several important questions unanswered such as the explicit structure of equations which could allow one to derive normal coordinate expressions for the connection and the vielbein to any order in perturbation theory (in this respect, the work [31], which closely follows the original normal coordinate construction of [27], contains very useful results). Our presentation in section 2 is based in part on earlier approaches developed in general relativity [34] and quantum gravity [35–38] many years ago, as well as some more recent covariant techniques for super Yang-Mills theories [39].⁶ Here we derive differential equations which are obeyed by the vielbein and the connection in normal coordinates, and which can be used to reconstruct these objects, in principle in closed form. We also present an equation for the super-determinant of the vielbein, $E = \text{Ber}(E_M^A)$, which allows one to reconstruct E without a detailed knowledge of the vielbein. As an application of the techniques developed in section 2, in section 3 we explicitly reduce an integral over the full 4D $\mathcal{N} = 2$ curved superspace to that over the chiral subspace.

Section 4 is central to the present work. Here we reduce the action (1.2) to components using the principle of projective invariance. We also consider two applications. First, we prove the gauge invariance of the special vector-tensor coupling introduced in [1]. Second, we give a curved superspace description for the c-map [41, 42]. In section 5, we derive a new representation for the covariantly chiral projector and use this result to reformulate the action (1.2) as a chiral integral.

This paper is accompanied by three technical appendices. In appendix A we collect the salient points of the superspace formulation for $\mathcal{N} = 2$ supergravity, following [1], which are essential for understanding the main results of this paper. Appendix B summarizes the main properties of covariant projective supermultiplets following [1]. Finally, appendix C provides the proof of eq. (5.1).

2 Integrating out fermionic dimensions

In this section, we temporarily leave aside the main object of our study – $\mathcal{N} = 2$ matter-coupled supergravity in four space-time dimensions, and instead discuss the problem of defining a normal coordinate system around a submanifold of a curved superspace with any number of bosonic and fermionic dimensions. We will present an application of the formalism developed to the case of 4D $\mathcal{N} = 2$ supergravity in section 3.

2.1 Parallel transport and associated two-point functions

Let us consider a curved superspace $\mathcal{M} \equiv \mathcal{M}^{d|\delta}$ with d space-time and δ fermionic dimensions, and let z^M be local coordinates chosen to parametrize \mathcal{M} . The corresponding superspace geometry is described by covariant derivatives

$$\mathcal{D}_A = E_A + \Phi_A, \quad E_A := E_A^M(z) \partial_M, \quad \Phi_A := \Phi_A(z) \cdot \mathbb{J} = E_A^M \Phi_M. \quad (2.1)$$

⁶The material in section 2 is based in part on unpublished lecture notes by one of us (SMK) [40].

Here \mathbb{J} denotes the generators of the structure group⁷ G (with all indices of \mathbb{J} s suppressed), E_A is the inverse vielbein, and $\Phi = dz^M \Phi_M = E^A \Phi_A$ the connection. As usual, the matrices defining the vielbein $E^A := dz^M E_M^A(z)$ and its inverse E_A obey the identities $E_A^M E_M^B = \delta_A^B$ and $E_M^A E_A^N = \delta_M^N$. An infinitesimal G -transformation acts on the components of a vector field $v = v^A E_A$ and a one-form $\omega = E^A \omega_A$ as follows:

$$[\lambda \cdot \mathbb{J}, v^A] = \lambda^A_B v^B = -v^B \lambda_B^A, \quad [\lambda \cdot \mathbb{J}, \omega_A] = -\omega_B \lambda^B_A = \lambda_A^B \omega_B, \quad (2.2)$$

such that $(v)\omega := v^A \omega_A$ is invariant. Here we have assumed that the structure group transformations preserve the Grassmann parity ε of any tensor superfield, which requires $\varepsilon(\lambda_A^B) = 0$, and the transformation parameters are defined to obey $\lambda_A^B = -\lambda^B_A$.

The covariant derivatives obey the algebra

$$[\mathcal{D}_A, \mathcal{D}_B] = T_{AB}^C \mathcal{D}_C + R_{AB} \cdot \mathbb{J}, \quad (2.3)$$

with T_{AB}^C the torsion, and R_{AB} the curvature of \mathcal{M} . In particular,

$$\{\mathcal{D}_A, \mathcal{D}_B\} \omega_C = T_{AB}^D \mathcal{D}_D \omega_C + R_{ABC}^D \omega_D, \quad (2.4)$$

when acting on the one-form ω_A .

It is pertinent to our consideration to recall the basic facts about parallel transport. Let $z' \in \mathcal{M}$ be a given superspace point, and $\gamma(t) = \{z^M(t)\}$ a smooth curve in \mathcal{M} such that $\gamma(0) = z'$. For the tangent vector to γ at $z(t)$, we convert its world index into a local flat one,

$$\zeta^A(t) := \dot{z}^M(t) E_M^A(z(t)). \quad (2.5)$$

Let $v^{A'} = v^{M'} E_{M'}^{A'}(z')$ be a tangent vector at z' , $v \in T_{z'} \mathcal{M}$. Its parallel transport along γ , $v(t) \in T_{z(t)} \mathcal{M}$, is defined to satisfy the equation

$$\left(\frac{d}{dt} + \zeta^B(t) \Phi_B(t) \right) v^A(t) = 0. \quad (2.6)$$

The parallel transport of a tensor \mathcal{V}' at z' along the curve $\gamma(t)$ is defined similarly.

All information about parallel transport along the curve $\gamma(t)$ is encoded in the corresponding *parallel displacement propagator along γ* , $I_\gamma(t) \in G$, which is defined by the following conditions:

- (i) the parallel transport equation

$$\left(\frac{d}{dt} + \zeta^B(t) \Phi_B(t) \right) I_\gamma(t) = 0; \quad (2.7)$$

- (ii) the initial condition

$$I_\gamma(0) = \mathbb{1}. \quad (2.8)$$

⁷The formalism below can be readily generalized to incorporate an internal Yang-Mills group G_{int} by replacing $G \rightarrow G \times G_{\text{int}}$.

Then, for any tensor \mathcal{V}' at z' , its parallel transport along $\gamma(t)$ is

$$\mathcal{V}(t) = D\left(I_\gamma(t)\right)\mathcal{V}', \tag{2.9}$$

where D is the representation of the structure group G in which the tensor transforms.⁸ As is known, a unique solution to eqs. (2.7) and (2.8) is the path-ordered exponential

$$I_\gamma(t) = \text{P e}^{-\int_\gamma \Phi}. \tag{2.10}$$

The important feature of the equation (2.7) is its invariance under reparametrizations of the curve.

Now, let $\hat{\gamma}(t) = \{z^M(t)\}$ be a geodesic through z' ,

$$\left(\frac{d}{dt} + \zeta^B(t)\Phi_B(t)\right)\zeta^A(t) = 0, \quad \hat{\gamma}(0) = z'. \tag{2.11}$$

For any point $z^M(t)$ on the geodesic, we define $I(z(t); z') := I_{\hat{\gamma}}(t)$. Since any two points z' and z in \mathcal{M} can be connected by a geodesic, which is locally unique modulo worldline reparametrizations, we obtain a well-defined two-point function

$$I(z; z') \in G, \quad I(z'; z') = \mathbb{1}. \tag{2.12}$$

It will be called the parallel displacement propagator.

The freedom to choose affine parametrization of the geodesic, which connects z' and z , can be fixed as

$$z' = \hat{\gamma}(0), \quad z = \hat{\gamma}(1), \tag{2.13}$$

which corresponds to the standard exponential mapping (see, e.g., [43]). For this parametrization, we define vector two-point functions⁹

$$\zeta^A(z; z') := \zeta^A(t=1) \in T_z\mathcal{M}, \tag{2.14a}$$

$$\zeta^{A'}(z'; z) := -\zeta^A(t=0) \in T_{z'}\mathcal{M}. \tag{2.14b}$$

These functions are related to each other as follows:

$$\zeta^A(z; z') = -[I(z; z')]^A_{B'} \zeta^{B'}(z'; z). \tag{2.15}$$

The parallel displacement propagator, $I(z; z')$, obeys the differential equations:

$$\zeta^B \mathcal{D}_B I(z; z') = 0, \tag{2.16a}$$

$$\zeta^{B'} \mathcal{D}_{B'} I(z; z') = 0. \tag{2.16b}$$

⁸In what follows, we do not indicate explicitly the representation D of the structure group, and the matrix $D(I_\gamma(t))$ will always be written simply as $I_\gamma(t)$.

⁹In the case when \mathcal{M} is an ordinary Riemannian manifold, in particular if $T_{AB}{}^C = 0$, one can show that $\zeta^A(z, z') = \mathcal{D}^A \sigma(z, z')$ and $\zeta^{A'}(z'; z) = \mathcal{D}^{A'} \sigma(z, z')$, where $\sigma(z, z') = \sigma(z', z)$ is the so-called *world function* coinciding with half the square of the geodesic distance between the points z' and z , see [34–36] for more detail. In the mathematics literature, the $\sigma(z, z')$ is sometimes referred to as the *distance function* [43].

These equations follow from (2.7). It also holds that

$$I(z; z') I(z'; z) = \mathbb{1} . \tag{2.17}$$

As to the two-point functions $\zeta^A(z; z')$ and $\zeta^{A'}(z'; z)$, they enjoy the following equations:

$$\zeta^B \mathcal{D}_B \zeta^A = \zeta^A , \tag{2.18a}$$

$$\zeta^B \mathcal{D}_B \zeta^{A'} = \zeta^{A'} . \tag{2.18b}$$

To prove eq. (2.18a), it suffices to note that for a geodesic $z^M(t)$ passing through z' , $z(0) = z'$, we have

$$\zeta^A(z(t); z') = t \zeta^A(t) , \tag{2.19}$$

with $\zeta^A(t)$ the tangent vector to the given geodesic at $z(t)$. Then, it only remains to use the geodesic equation (2.11). As to equation (2.18b), it now follows from the relations (2.15), (2.16a) and (2.18a).

2.2 Covariant Taylor expansion

Let $\mathcal{V}(z)$ be a tensor superfield transforming in some representation of the structure group. Then it can be expanded in a covariant Taylor series of the form:

$$I(z'; z) \mathcal{V}(z) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \zeta^{A'_n} \dots \zeta^{A'_2} \zeta^{A'_1} \mathcal{D}_{A'_1} \mathcal{D}_{A'_2} \dots \mathcal{D}_{A'_n} \mathcal{V}(z') . \tag{2.20}$$

It can be justified simply by generalizing the proof given, e.g., in [37] for the case when \mathcal{M} is a Riemannian manifold.

2.3 Parallel transport around the submanifold

Up to now, we have considered all possible geodesics passing through a fixed point $z' \in \mathcal{M}$, where the latter have been completely arbitrary. From now on, we turn to a more general setup. First of all, we will restrict z' to belong to a fixed submanifold $\Sigma \equiv \Sigma^{d'|\delta'}$ of the superspace $\mathcal{M} = \mathcal{M}^{d|\delta}$, with $\delta' < \delta$ or/and $d' < d$. Secondly, we will only consider those geodesics $\hat{\gamma}(t)$ through z' , $\hat{\gamma}(0) = z'$, which are transverse to Σ . To make the latter requirement more precise, we assume in addition that the vielbein E^A s can be split into two disjoint subsets,

$$E^A = (E^{\hat{a}}, E^{\hat{\alpha}}) , \tag{2.21}$$

such that the set of one-forms $E^{\hat{a}}|_{z'}$ constitutes a basis of the cotangent space $T_{z'}^* \Sigma$ at any point $z' \in \Sigma$. Then, the requirement that $\hat{\gamma}(t)$ be transverse to Σ , will mean the following:

$$\dot{z}^M(0) E_M^{\hat{a}}(z') = 0 , \quad z(0) = z' \in \Sigma . \tag{2.22}$$

Finally, we put forward one more technical requirement, that the structure group G acts reducibly on E^A s such that each of the two subsets $E^{\hat{a}}$ s and $E^{\hat{\alpha}}$ s transforms into itself under

the action of G . The setup introduced here reduces to that considered in subsection 2.1 if Σ shrinks down to a single point z' .

Let $\tilde{z}^{\hat{m}}$ be local coordinates parametrizing the submanifold Σ . These variables can be extended to provide a local coordinate system $z^M = (\tilde{z}^{\hat{m}}, y^{\hat{\mu}})$ in the whole superspace \mathcal{M} in such a way that along Σ we have

$$z^M|_{\Sigma} = (\tilde{z}^{\hat{m}}, y^{\hat{\mu}} = 0) . \quad (2.23)$$

Reparametrization invariance can be further used to choose

$$E_M{}^A(z)|_{\Sigma} = \begin{pmatrix} \mathcal{E}_{\hat{m}}^{\hat{a}}(\tilde{z}) & \mathcal{E}_{\hat{m}}^{\hat{\alpha}}(\tilde{z}) \\ 0 & \delta_{\hat{\mu}}^{\hat{\alpha}} \end{pmatrix} . \quad (2.24)$$

Then, eq. (2.22) becomes

$$\dot{z}^M(0) = (0, \dot{y}^{\hat{\mu}}(0)) . \quad (2.25)$$

In terms of $\zeta^A(t)$, eq. (2.5), this is equivalent to

$$\zeta^A(0) = \zeta^{\hat{\mu}} \delta_{\hat{\mu}}^A , \quad \zeta^{\hat{\mu}} \equiv \dot{y}^{\hat{\mu}}(0) . \quad (2.26)$$

It follows from the above consideration that

$$\zeta^{\hat{a}}(z; z') = \zeta^{\hat{a}'}(z'; z) = 0 . \quad (2.27)$$

As an example, let us consider a curved superspace corresponding to four-dimensional $\mathcal{N} = 2$ conformal supergravity reviewed in appendix A. It follows from the anticommutation relations (A.9b) that the vector fields¹⁰ $E_{\hat{\alpha}} := \bar{E}_i^{\hat{\alpha}}$ generate an involutive distribution (see, e.g., [43] for a review of the relevant differential-geometric constructions), that is

$$\{\bar{E}_i^{\hat{\alpha}}, \bar{E}_j^{\hat{\beta}}\} = C_i^{\hat{\alpha}}{}_{\hat{\gamma}}{}^{\hat{\beta}}{}^k(z) \bar{E}_k^{\hat{\gamma}} . \quad (2.28)$$

Then, the Frobenius theorem (see, e.g., [43]) implies that one can replace the original local coordinates z^M by new ones, $\{\tilde{z}^{\hat{m}}, \rho^{\hat{\mu}}\}$, with the properties:

$$E_{\hat{\alpha}} \tilde{z}^{\hat{m}} = 0, \quad E_{\hat{\alpha}} = N_{\hat{\alpha}}^{\hat{\mu}}(\tilde{z}, \rho) \frac{\partial}{\partial \rho^{\hat{\mu}}} , \quad (2.29)$$

for some non-singular matrix $N_{\hat{\alpha}}^{\hat{\mu}}$. It is clear that covariantly chiral scalar superfields, $\bar{\mathcal{D}}_i^{\hat{\alpha}} \Phi = 0$, are functions of the variables $\tilde{z}^{\hat{m}}, \Phi = \Phi(\tilde{z})$. The submanifold Σ in the above discussion will be identified with the chiral subspace defined by the equations $\rho^{\hat{\mu}} = 0$. Replacing $\rho^{\hat{\mu}}$ by new variables $y^{\hat{\mu}}$ defined as

$$\rho^{\hat{\mu}} = y^{\hat{\nu}} \delta_{\hat{\nu}}^{\hat{\alpha}} N_{\hat{\alpha}}^{\hat{\mu}}(\tilde{z}, \rho) , \quad (2.30)$$

one can see that the inverse vielbein restricted to Σ has the form:

$$E_A{}^M(z)|_{\Sigma} = \begin{pmatrix} \mathcal{E}_{\hat{a}}^{\hat{m}}(\tilde{z}) & \mathcal{E}_{\hat{a}}^{\hat{\mu}}(\tilde{z}) \\ 0 & \delta_{\hat{\alpha}}^{\hat{\mu}} \end{pmatrix} . \quad (2.31)$$

This result is equivalent to (2.24). In the example considered, the involutive distribution generated by $\bar{E}_i^{\hat{\alpha}}$, determines all the tangent vectors being transverse to Σ .

¹⁰The inverse vielbein is thus $E_A = (E_{\hat{a}}, E_{\hat{\alpha}})$, where $E_{\hat{a}} := (E_a, E_{\alpha}^i)$.

2.4 Normal coordinates around the submanifold

A normal coordinate system¹¹ around Σ is defined by the following two conditions:

- (i) All geodesics, which are transverse to Σ , are straight lines.

$$\tilde{z}^{\hat{m}}(t) = \tilde{z}^{\hat{m}}, \quad y^{\hat{\mu}}(t) = t \zeta^{\hat{\mu}}. \quad (2.32)$$

Such a geodesic connects the superspace points $(\tilde{z}, 0)$ and (\tilde{z}, ζ) .

- (ii) Fock-Schwinger (or structure group) gauge:

$$I(z; z') = I(\tilde{z}, \zeta; \tilde{z}, 0) = \mathbb{1}. \quad (2.33)$$

For the two-point function $\zeta^A(z, z')$, eq. (2.14a), the condition (2.32) implies

$$\zeta^A(z; z') = \zeta^{\hat{\mu}} E_{\hat{\mu}}^A(\tilde{z}, \zeta) \equiv \zeta^M E_M^A(\tilde{z}, \zeta), \quad \zeta^M := (0, \zeta^{\hat{\mu}}). \quad (2.34)$$

For the two-point function $\zeta^{A'}(z'; z)$, eq. (2.14b), the condition (2.26) gives

$$\zeta^{A'}(z'; z) = -\zeta^M \delta_M^{A'}. \quad (2.35)$$

Now, using eqs. (2.15) and (2.33) gives

$$\zeta^M E_M^A(\tilde{z}, \zeta) = \zeta^M \delta_M^A = \zeta^{\hat{\mu}} \delta_{\hat{\mu}}^{\hat{\alpha}}. \quad (2.36)$$

Furthermore, using eqs. (2.16a) and (2.33) gives

$$\zeta^A \Phi_A(\tilde{z}, \zeta) \cdot \mathbb{J} = \zeta^M \Phi_M(\tilde{z}, \zeta) \cdot \mathbb{J} = 0. \quad (2.37)$$

The relations (2.36) and (2.37) are the key results for applications.¹² These relations did not appear in [15]. It is worth pointing out that eq. (2.37) implies

$$\Phi_{\hat{\alpha}}(\tilde{z}, 0) \cdot \mathbb{J} = \Phi_{\hat{\mu}}(\tilde{z}, 0) \cdot \mathbb{J} = 0, \quad (2.38)$$

while no restriction is imposed on $\Phi_{\hat{m}}(\tilde{z}, 0) \cdot \mathbb{J}$ which is the connection on Σ .

Relations (2.36) and (2.37) can be rewritten in terms of the operation of interior product, ι_{ζ} . It is worth recalling how the latter is defined. Given a vector field $\mathcal{V} = \mathcal{V}^M \partial_M = \mathcal{V}^A E_A$ and a p -form

$$\Omega = \frac{1}{p!} dz^{M_p} \dots dz^{M_1} \Omega_{M_1 \dots M_p} = \frac{1}{p!} E^{A_p} \dots E^{A_1} \Omega_{A_1 \dots A_p}, \quad (2.39)$$

the $(p-1)$ -form $\iota_{\mathcal{V}} \Omega$ is defined as

$$\iota_{\mathcal{V}} \Omega = \frac{1}{(p-1)!} dz^{M_p} \dots dz^{M_2} \mathcal{V}^{M_1} \Omega_{M_1 \dots M_p} = \frac{1}{(p-1)!} E^{A_p} \dots E^{A_2} \mathcal{V}^{A_1} \Omega_{A_1 \dots A_p}. \quad (2.40)$$

Now, eqs. (2.36) and (2.37) can be rewritten as follows:

$$\iota_{\zeta} E^A = \zeta^M \delta_M^A = \zeta^{\hat{\mu}} \delta_{\hat{\mu}}^{\hat{\alpha}}, \quad (2.41a)$$

$$\iota_{\zeta} \Phi_A^B = 0, \quad (2.41b)$$

with $\Phi_A^B = dz^M \Phi_{MA}^B = E^C \Phi_{CA}^B$ the connection one-form.

¹¹In Riemannian geometry, normal coordinates around a submanifold were discussed in [44].

¹²In the zero-dimensional case when Σ reduces to a single point z' , the relations (2.36) and (2.37) are equivalent to those given in [27]. In the case when $\Sigma = \Sigma^{(d,0)}$ is the bosonic body of the curved superspace $\mathcal{M} = \mathcal{M}^{(d,\delta)}$, the relations (2.36) and (2.37) were derived in [31] in a different manner.

2.5 Structure equations

We turn to uncovering the implications of eqs. (2.41a) and (2.41b), building on the construction in Riemannian geometry given in [45, 46].

We start by introducing the torsion two-form

$$T^A = \frac{1}{2} E^C E^B T_{BC}{}^A \quad (2.42)$$

and the curvature two-form

$$R \cdot \mathbb{J} = \frac{1}{2} E^D E^C R_{CD} \cdot \mathbb{J}, \quad [R \cdot \mathbb{J}, \omega_A] = R_A{}^B \omega_B = \frac{1}{2} E^D E^C R_{CDA}{}^B \omega_B, \quad (2.43)$$

with ω_A an arbitrary one-form. They obey the structure equations:

$$-T^A = dE^A - E^B \Phi_B{}^A, \quad (2.44a)$$

$$R_A{}^B = d\Phi_A{}^B - \Phi_A{}^C \Phi_C{}^B. \quad (2.44b)$$

Let us make use of the well-known differential geometric relation

$$L_\zeta = \iota_\zeta d + d \iota_\zeta, \quad (2.45)$$

with L_ζ the Lie derivative. Applying both sides of this relation to $\Phi_A{}^B$ and using the structure equation (2.44b) and the gauge condition (2.41b), we obtain

$$L_\zeta \Phi_A{}^B = \iota_\zeta R_A{}^B. \quad (2.46)$$

Similarly we can evaluate $L_\zeta E^A$ to obtain

$$L_\zeta E^A = \mathcal{D}\zeta^A - \iota_\zeta T^A, \quad \mathcal{D}\zeta^A := d\zeta^A - \zeta^B \Phi_B{}^A. \quad (2.47)$$

Applying again L_ζ to both sides of (2.47) and making use of the gauge conditions and the structure equations, one obtains

$$(L_\zeta - 1)L_\zeta E^A = -\mathcal{D}\zeta^D \zeta^C T_{CD}{}^A + (\iota_\zeta T^D) \zeta^C T_{CD}{}^A - E^D \zeta^C L_\zeta T_{CD}{}^A - \zeta^B \iota_\zeta R_B{}^A. \quad (2.48)$$

Here the Lie derivative of the torsion tensor can be represented, due to (2.37), as

$$L_\zeta T_{CD}{}^A = \zeta^{\hat{\nu}} \partial_{\hat{\nu}} T_{CD}{}^A = \zeta^{\hat{\beta}} \mathcal{D}_{\hat{\beta}} T_{CD}{}^A. \quad (2.49)$$

The Lie derivative of a one-form is

$$L_{\mathcal{V}} \omega_M = \mathcal{V}^N \partial_N \omega_M + \left(\frac{\partial}{\partial z^M} \mathcal{V}^N \right) \omega_N, \quad (2.50)$$

and thus

$$L_\zeta \omega_M = \zeta^{\hat{\nu}} \partial_{\hat{\nu}} \omega_M + \delta_M{}^{\hat{\nu}} \omega_{\hat{\nu}} \implies \begin{cases} L_\zeta \omega_{\hat{m}} = \zeta \cdot \partial \omega_{\hat{m}} \\ L_\zeta \omega_{\hat{\mu}} = (\zeta \cdot \partial + 1) \omega_{\hat{\mu}} \end{cases} \quad (2.51)$$

The relations (2.46) and (2.47), and their corollary (2.48), allow us to reconstruct the connection $\Phi_{MA}{}^B(\tilde{z}, \zeta)$ and the vielbein $E_M{}^A(\tilde{z}, \zeta)$ as Taylor series in ζ s, in which all the coefficients (except the leading ζ -independent terms) are tensor functions of the torsion, the curvature and their covariant derivatives evaluated at $\zeta = 0$ (of course, there also occur contributions involving the field $\mathcal{E}_{\hat{m}}{}^{\hat{a}}(\tilde{z})$ defined in (2.24)). Indeed, consider a tensor superfield \mathcal{V} such as $T_{CD}{}^A$ or $R_{CDB}{}^A$ and their covariant derivatives. In the normal gauge, the covariant Taylor expansion, eq. (2.20), becomes

$$\mathcal{V}(\tilde{z}, \zeta) = \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^{\hat{\alpha}_n} \dots \zeta^{\hat{\alpha}_1} \mathcal{D}_{\hat{\alpha}_1} \dots \mathcal{D}_{\hat{\alpha}_n} \mathcal{V}(\tilde{z}, 0) \equiv \sum_{n=0}^{\infty} \mathcal{V}^{(n)}, \quad \zeta \cdot \partial \mathcal{V}^{(n)} = n \mathcal{V}^{(n)}, \quad (2.52)$$

with $\zeta^{\hat{\alpha}} \equiv \zeta^{\hat{\mu}} \delta_{\hat{\mu}}{}^{\hat{\alpha}}$. Eq. (2.46) can be rewritten in the component form:

$$L_{\zeta} \Phi_{MA}{}^B(\tilde{z}, \zeta) = E_M{}^D(\tilde{z}, \zeta) \zeta^{\hat{\gamma}} R_{\hat{\gamma}DA}{}^B(\tilde{z}, \zeta), \quad (2.53)$$

and similarly for eq. (2.47) or its corollary (2.48). Now, all tensors involved have to be represented by covariant Taylor series of the form (2.52), while $\Phi_{MA}{}^B(\tilde{z}, \zeta)$ and $E_M{}^A(\tilde{z}, \zeta)$ have to be given as ordinary Taylor series, in particular

$$E_M{}^A(\tilde{z}, \zeta) = \sum_{n=0}^{\infty} \frac{1}{n!} \zeta^{\hat{\nu}_n} \dots \zeta^{\hat{\nu}_1} \partial_{\hat{\nu}_1} \dots \partial_{\hat{\nu}_n} E_M{}^A(\tilde{z}, 0) \equiv \sum_{n=0}^{\infty} E^{(n)}{}_M{}^A. \quad (2.54)$$

In accordance with (2.51), the Lie derivative L_{ζ} acts on $E^{(n)}{}_M{}^A$ in (2.54), which is homogeneous of n -th degree in ζ , as the operator of multiplication by n if $M = \hat{m}$ or by $(n+1)$ if $M = \hat{\mu}$.

2.6 Computing the determinant of the vielbein

Of crucial importance is the explicit ζ -dependence of the determinant $E := \text{Ber}(E_M{}^A)$. The simplest way to address this problem is to derive a differential equation obeyed by E that follows from the equations given in the previous section.

Using the standard identity $\delta E = (-1)^M E \delta E_M{}^A E_A{}^M$ in conjunction with eq. (2.51), we obtain

$$\zeta \cdot \partial \ln E = (-1)^M [L_{\zeta} E_M{}^A - \delta_M{}^{\hat{\nu}} E_{\hat{\nu}}{}^A] E_A{}^M. \quad (2.55)$$

The right-hand side here can be transformed using the structure equation (2.47) to get

$$\zeta \cdot \partial \ln E = -(-1)^A [\Phi_{A\hat{\beta}}{}^A \zeta^{\hat{\beta}} + \zeta^{\hat{\beta}} T_{\hat{\beta}A}{}^A] + (-1)^{\hat{\mu}} \delta_{\hat{\mu}}{}^{\hat{\alpha}} (E_{\hat{\alpha}}{}^{\hat{\mu}} - \delta_{\hat{\alpha}}{}^{\hat{\mu}}). \quad (2.56)$$

This is the master equation to determine the ζ -dependence of $E = E(\tilde{z}, \zeta)$ under the boundary condition $E(\tilde{z}, 0) = \mathcal{E}(\tilde{z})$, where $\mathcal{E} = \text{Ber}(\mathcal{E}_{\hat{m}}{}^{\hat{a}})$ is the determinant of the vielbein on the submanifold, as introduced in eq. (2.24). Eq. (2.56) shows that one has to know the ζ -dependence of the connection in order to evaluate that of E . This result is quite nice and, at the same time, somewhat counter-intuitive, for one usually evaluates the vielbein only, while the explicit structure of the connection is completely ignored. For instance, the authors of [15] use a more laborious approach, which is:

- (i) to compute the ζ -dependence of the vielbein $E_M{}^A$ by iterations; and then
- (ii) to evaluate the determinant of the vielbein.

Equation (2.56) can be rewritten in a somewhat different form if one recalls that the structure group has been assumed to act reducibly, that is $\Phi_{A\hat{\beta}}{}^C = \Phi_{A\hat{\beta}}{}^{\hat{\gamma}} \delta_{\hat{\gamma}}{}^C$. This gives

$$\zeta \cdot \partial \ln E = -(-1)^{\hat{\alpha}} \Phi_{\hat{\alpha}\hat{\beta}}{}^{\hat{\alpha}} \zeta^{\hat{\beta}} - (-1)^A \zeta^{\hat{\beta}} T_{\hat{\beta}A}{}^A + (-1)^{\hat{\mu}} \delta_{\hat{\mu}}{}^{\hat{\alpha}} (E_{\hat{\alpha}}{}^{\hat{\mu}} - \delta_{\hat{\alpha}}{}^{\hat{\mu}}) . \quad (2.57)$$

It often happens that

$$(-1)^A T_{\hat{\beta}A}{}^A = 0 . \quad (2.58)$$

In particular, such a situation occurs in the cases of $\mathcal{N} = 1$ and $\mathcal{N} = 2$ supergravity when $\zeta^{\hat{\alpha}}$ are Grassmann coordinates. In this case we end up with the remarkably simple equation:

$$\zeta \cdot \partial \ln E = -(-1)^{\hat{\alpha}} \Phi_{\hat{\alpha}\hat{\beta}}{}^{\hat{\alpha}} \zeta^{\hat{\beta}} + (-1)^{\hat{\mu}} \delta_{\hat{\mu}}{}^{\hat{\alpha}} (E_{\hat{\alpha}}{}^{\hat{\mu}} - \delta_{\hat{\alpha}}{}^{\hat{\mu}}) . \quad (2.59)$$

3 Reduction to chiral subspace in $\mathcal{N} = 2$ supergravity

As an illustration of the normal coordinate techniques developed in section 2, here we apply the scheme to the case when \mathcal{M} is the curved 4D $\mathcal{N} = 2$ superspace as defined in appendix A, and Σ its chiral subspace. All the relevant information regarding the chiral subspace can be found at the end of subsection 2.3. Our goal is to reduce an integral over the full superspace, $\int d^4x d^4\theta d^4\bar{\theta} E U$, to that over the chiral subspace, for any scalar and isoscalar superfield U .

In this section we continue to use the ‘‘hat’’ index notation, which was introduced in section 2, as much as possible, keeping in mind that, for instance, $\mathcal{D}_{\hat{\alpha}} := \bar{\mathcal{D}}_i^{\hat{\alpha}}$. We also use the notation (2.52), with $\mathcal{V}^{(n)}$ denoting the n -th level of the ζ -expansion of \mathcal{V} . Moreover, one more piece of notation used throughout this section is the following: given a superfield $U(z)$, we denote $U| = U(z)|_{\Sigma}$ to be its projection to the chiral superspace.

We focus on the computation of E using equation (2.59) which in our case becomes

$$\zeta \cdot \partial \ln E = E_{\hat{\alpha}}{}^{\hat{\mu}} \Phi_{\hat{\mu}\hat{\beta}}{}^{\hat{\alpha}} \zeta^{\hat{\beta}} - \delta_{\hat{\mu}}{}^{\hat{\alpha}} (E_{\hat{\alpha}}{}^{\hat{\mu}} - \delta_{\hat{\alpha}}{}^{\hat{\mu}}) . \quad (3.1)$$

One should bear in mind that the connection now includes both the Lorentz and SU(2) terms, see appendix A. To determine the right hand side of (3.1) one needs to know special components of the connection, the vielbein and its inverse as functions of ζ . These can be found by solving iteratively, order-by-order in powers of ζ , the equations¹³ (2.46)–(2.48).

One can notice several important simplifications even before starting to solve eqs. (2.46)–(2.48). First of all, equation (2.29) tells us that

$$E_{\hat{\mu}}{}^{\hat{\alpha}} = E_{\hat{\alpha}}{}^{\hat{\mu}} = 0 . \quad (3.2)$$

¹³Equation (2.47) has to be used instead of (2.48) in order to determine the vielbein at first order in ζ . This follows from the fact that $(L_{\zeta} - 1)L_{\zeta} E^{(1)}{}_{\hat{m}}{}^A = 0$.

Second, since the structure group does not mix up the one-forms $E^{\hat{a}}$ and $E^{\hat{\alpha}}$, the following identities hold: $\Phi_{\hat{a}}^{\hat{\beta}} = \Phi_{\hat{\alpha}}^{\hat{b}} = R_{\hat{a}}^{\hat{\beta}} = R_{\hat{\alpha}}^{\hat{b}} = 0$. These results imply that eqs. (2.46)–(2.48) allow one to evaluate $E_{\hat{\mu}}^{\hat{\alpha}}$, $E_{\hat{\alpha}}^{\hat{\mu}}$ and $\Phi_{\hat{\mu}\hat{\alpha}}^{\hat{\beta}}$ without knowing the other components of E_M^A , E_A^M and Φ_{MA}^B .

Let us turn to evaluating $E_{\hat{\mu}}^{\hat{\alpha}}$ and $\Phi_{\hat{\mu}\hat{\alpha}}^{\hat{\beta}}$ using eqs. (2.46)–(2.48). According to the definition of the normal coordinate system, we have

$$E_{\hat{\mu}}^{\hat{\alpha}}| = \delta_{\hat{\mu}}^{\hat{\alpha}}, \quad E_{\hat{\alpha}}^{\hat{\mu}}| = \delta_{\hat{\alpha}}^{\hat{\mu}}, \quad \Phi_{\hat{\mu}\hat{\alpha}}^{\hat{\beta}}| = 0. \quad (3.3)$$

Since $T_{\hat{\gamma}\hat{\beta}}^{\hat{\alpha}} = 0$, equation (2.47) implies that

$$E^{(1)}_{\hat{\mu}}{}^{\hat{\alpha}} = 0. \quad (3.4)$$

Next, equation (2.46) has the following consequence:

$$(\zeta \cdot \partial + 1)\Phi_{\hat{\mu}\hat{\alpha}}^{\hat{\beta}} = E_{\hat{\mu}}^{\hat{\delta}}\zeta^{\hat{\gamma}}R_{\hat{\gamma}\hat{\delta}\hat{\alpha}}^{\hat{\beta}}. \quad (3.5)$$

To first order in ζ , the latter gives

$$\Phi^{(1)}_{\hat{\mu}\hat{\alpha}}{}^{\hat{\beta}} = \frac{1}{2}\delta_{\hat{\mu}}^{\hat{\delta}}R_{\hat{\delta}\hat{\gamma}\hat{\alpha}}^{\hat{\beta}}|\zeta^{\hat{\gamma}}. \quad (3.6)$$

To compute $E_{\hat{\mu}}^{\hat{\alpha}}$ to second order in ζ , it is handy to use equation (2.48) which gives

$$E^{(2)}_{\hat{\mu}}{}^{\hat{\alpha}} = \frac{1}{6}\delta_{\hat{\mu}}^{\hat{\delta}}R_{\hat{\delta}\hat{\gamma}\hat{\beta}}^{\hat{\alpha}}|\zeta^{\hat{\beta}}\zeta^{\hat{\gamma}}. \quad (3.7)$$

Next, making use of (3.4) and (3.5) gives

$$\Phi^{(2)}_{\hat{\mu}\hat{\alpha}}{}^{\hat{\beta}} = \frac{1}{3}\delta_{\hat{\mu}}^{\hat{\delta}}(\mathcal{D}_{\hat{\rho}}R_{\hat{\delta}\hat{\gamma}\hat{\alpha}}^{\hat{\beta}})|\zeta^{\hat{\gamma}}\zeta^{\hat{\rho}}. \quad (3.8)$$

Here we have used, for the first time, the covariant Taylor expansion (2.52) of the curvature. Further iterations lead to

$$E^{(3)}_{\hat{\mu}}{}^{\hat{\alpha}} = -\frac{1}{12}\delta_{\hat{\mu}}^{\hat{\delta}}(\mathcal{D}_{\hat{\rho}}R_{\hat{\delta}\hat{\gamma}\hat{\beta}}^{\hat{\alpha}})|\zeta^{\hat{\beta}}\zeta^{\hat{\gamma}}\zeta^{\hat{\rho}}, \quad (3.9a)$$

$$\Phi^{(3)}_{\hat{\mu}\hat{\alpha}}{}^{\hat{\beta}} = \frac{1}{8}\delta_{\hat{\mu}}^{\hat{\tau}}\left(\frac{1}{3}R_{\hat{\tau}\hat{\rho}\hat{\delta}}^{\hat{\delta}'}R_{\hat{\delta}'\hat{\gamma}\hat{\alpha}}^{\hat{\beta}} + (\mathcal{D}_{\hat{\rho}}\mathcal{D}_{\hat{\delta}}R_{\hat{\tau}\hat{\gamma}\hat{\alpha}}^{\hat{\beta}})\right)|\zeta^{\hat{\gamma}}\zeta^{\hat{\delta}}\zeta^{\hat{\rho}}, \quad (3.9b)$$

$$E^{(4)}_{\hat{\mu}}{}^{\hat{\alpha}} = \frac{1}{20}\delta_{\hat{\mu}}^{\hat{\beta}}\left(\frac{1}{6}R_{\hat{\beta}\hat{\tau}\hat{\rho}}^{\hat{\delta}'}R_{\hat{\delta}'\hat{\delta}\hat{\gamma}}^{\hat{\alpha}} + (\mathcal{D}_{\hat{\tau}}\mathcal{D}_{\hat{\rho}}R_{\hat{\beta}\hat{\delta}\hat{\gamma}}^{\hat{\alpha}})\right)|\zeta^{\hat{\gamma}}\zeta^{\hat{\delta}}\zeta^{\hat{\rho}}\zeta^{\hat{\tau}}. \quad (3.9c)$$

As a result, we have computed the components $E_{\hat{\mu}}^{\hat{\alpha}}$ of the vielbein,

$$E_{\hat{\mu}}^{\hat{\alpha}} = \delta_{\hat{\mu}}^{\hat{\alpha}} + E^{(2)}_{\hat{\mu}}{}^{\hat{\alpha}} + E^{(3)}_{\hat{\mu}}{}^{\hat{\alpha}} + E^{(4)}_{\hat{\mu}}{}^{\hat{\alpha}}. \quad (3.10)$$

Since $E_{\hat{\mu}}^{\hat{a}} = 0$, the components $E_{\hat{\alpha}}^{\hat{\mu}}$ of the inverse vielbein constitute the inverse of the matrix (3.10) which can be easily computed. Now, the master equation (3.1) becomes

$$\begin{aligned} \zeta \cdot \partial \ln E &= \delta_{\hat{\alpha}}^{\hat{\mu}}\Phi^{(1)}_{\hat{\mu}\hat{\beta}}{}^{\hat{\alpha}}\zeta^{\hat{\beta}} + \delta_{\hat{\alpha}}^{\hat{\mu}}\Phi^{(2)}_{\hat{\mu}\hat{\beta}}{}^{\hat{\alpha}}\zeta^{\hat{\beta}} + \delta_{\hat{\alpha}}^{\hat{\mu}}\Phi^{(3)}_{\hat{\mu}\hat{\beta}}{}^{\hat{\alpha}}\zeta^{\hat{\beta}} - \delta_{\hat{\alpha}}^{\hat{\nu}}\delta_{\hat{\gamma}}^{\hat{\mu}}E^{(2)}_{\hat{\nu}}{}^{\hat{\gamma}}\Phi^{(1)}_{\hat{\mu}\hat{\beta}}{}^{\hat{\alpha}}\zeta^{\hat{\beta}} \\ &+ \delta_{\hat{\alpha}}^{\hat{\mu}}E^{(2)}_{\hat{\mu}}{}^{\hat{\alpha}} - \delta_{\hat{\alpha}}^{\hat{\mu}}\delta_{\hat{\gamma}}^{\hat{\nu}}E^{(2)}_{\hat{\mu}}{}^{\hat{\gamma}}E^{(2)}_{\hat{\nu}}{}^{\hat{\alpha}} + \delta_{\hat{\alpha}}^{\hat{\mu}}E^{(3)}_{\hat{\mu}}{}^{\hat{\alpha}} + \delta_{\hat{\alpha}}^{\hat{\mu}}E^{(4)}_{\hat{\mu}}{}^{\hat{\alpha}}. \end{aligned} \quad (3.11)$$

At this stage, we need the explicit form of the curvature $R_{\hat{\alpha}\hat{\beta}\hat{\gamma}}^{\hat{\delta}}$. In accordance with (2.4), it can be read off from the anticommutator $\{\bar{\mathcal{D}}_i^{\hat{\alpha}}, \bar{\mathcal{D}}_j^{\hat{\beta}}\}$, eq. (A.9b).

$$\begin{aligned} R_{\hat{\alpha}\hat{\beta}\hat{\gamma}}^{\hat{\delta}} &= R_{i j k \hat{\delta}}^{\hat{\alpha} \hat{\beta} \hat{\gamma} l} \\ &= \left(4\bar{S}_{ij}\varepsilon^{\hat{\gamma}(\hat{\alpha}}\delta_{\hat{\delta}}^{\hat{\beta})}\delta_k^l + 2\varepsilon_{ij}\varepsilon^{\hat{\alpha}\hat{\beta}}\bar{Y}^{\hat{\gamma}}_{\hat{\delta}}\delta_k^l + 2\varepsilon_{ij}\varepsilon^{\hat{\alpha}\hat{\beta}}\bar{S}_k^l\delta_{\hat{\delta}}^{\hat{\gamma}} + 4\bar{Y}^{\hat{\alpha}\hat{\beta}}\varepsilon_{k(i}\delta_{j)}^l\delta_{\hat{\delta}}^{\hat{\gamma}} \right), \end{aligned} \quad (3.12)$$

and hence

$$R_{\hat{\alpha}\hat{\beta}\hat{\gamma}}^{\hat{\alpha}} = -4\bar{S}_{jk}\varepsilon^{\hat{\beta}\hat{\gamma}} - 4\bar{Y}^{\hat{\beta}\hat{\gamma}}\varepsilon_{jk}. \quad (3.13)$$

Now, using (3.13), the relations

$$\zeta^{\hat{\alpha}}\zeta^{\hat{\beta}} = \frac{1}{2}(\varepsilon^{ij}\zeta_{\hat{\alpha}\hat{\beta}} - \varepsilon_{\hat{\alpha}\hat{\beta}}\zeta^{ij}), \quad \zeta_{\hat{\alpha}\hat{\beta}} := \zeta_{\hat{\alpha}k}\zeta_{\hat{\beta}}^k = \zeta_{\hat{\beta}\hat{\alpha}}, \quad \zeta^{ij} := \zeta_{\hat{\gamma}}^i\zeta^{\hat{\gamma}j} = \zeta^{ji}, \quad (3.14)$$

$$\zeta^{\hat{\alpha}}\zeta^{\hat{\beta}}\zeta^{\hat{\gamma}} = \frac{1}{3}\varepsilon^{jkl}\varepsilon_{\hat{\alpha}(\hat{\beta}}\zeta_{\hat{\gamma})q}\zeta^{iq} - \frac{1}{3}\varepsilon_{\hat{\beta}\hat{\gamma}}\varepsilon^{i(j}\zeta_{\hat{\alpha}q}\zeta^{k)q}, \quad (3.15)$$

and the Bianchi identities (A.12), one can prove that

$$\delta_{\hat{\alpha}}^{\hat{\mu}}E^{(3)}_{\hat{\mu}}^{\hat{\alpha}} = 0. \quad (3.16)$$

Then eq. (3.11) drastically simplifies

$$\zeta \cdot \partial \ln E = -\frac{1}{3}R_{\hat{\alpha}\hat{\gamma}\hat{\beta}}^{\hat{\alpha}}|\zeta^{\hat{\beta}}\zeta^{\hat{\gamma}} + \frac{1}{45}R_{\hat{\alpha}\hat{\tau}\hat{\rho}}^{\hat{\delta}}R_{\hat{\delta}\hat{\gamma}\hat{\beta}}^{\hat{\alpha}}|\zeta^{\hat{\beta}}\zeta^{\hat{\gamma}}\zeta^{\hat{\rho}}\zeta^{\hat{\tau}}. \quad (3.17)$$

Making use of the relations (3.12) and (3.13) along with the identities

$$\zeta^4 := \frac{1}{3}\zeta^{ij}\zeta_{ij}, \quad \zeta^{ij}\zeta^{kl} = -\varepsilon^{i(k}\varepsilon^{l)j}\zeta^4, \quad \zeta_{\hat{\alpha}\hat{\beta}}\zeta_{\hat{\gamma}\hat{\delta}} = \varepsilon_{\hat{\alpha}(\hat{\gamma}}\varepsilon_{\hat{\delta})\hat{\beta}}\zeta^4, \quad \zeta_{\hat{\alpha}\hat{\beta}}\zeta^{ij} = 0, \quad (3.18a)$$

$$\zeta^{\hat{\alpha}}\zeta^{\hat{\beta}}\zeta^{\hat{\gamma}}\zeta^{\hat{\delta}} = \frac{1}{4}(\varepsilon^{ij}\varepsilon^{kl}\varepsilon_{\hat{\alpha}(\hat{\gamma}}\varepsilon_{\hat{\delta})\hat{\beta}} - \varepsilon_{\hat{\alpha}\hat{\beta}}\varepsilon_{\hat{\gamma}\hat{\delta}}\varepsilon^{i(k}\varepsilon^{l)j})\zeta^4, \quad (3.18b)$$

equation (3.17) becomes

$$\zeta \cdot \partial \ln E = \frac{4}{3}\bar{Y}^{\hat{\alpha}\hat{\beta}}|\zeta_{\hat{\alpha}\hat{\beta}} - \frac{4}{3}\bar{S}_{ij}|\zeta^{ij} + \frac{8}{27}(\bar{Y}^{\hat{\alpha}\hat{\beta}}\bar{Y}_{\hat{\alpha}\hat{\beta}} - \bar{S}_{ij}\bar{S}^{ij})|\zeta^4. \quad (3.19)$$

Its solution is given by the simple formula

$$E = \mathcal{E} \left(1 + \frac{2}{3}\bar{Y}^{\hat{\alpha}\hat{\beta}}|\zeta_{\hat{\alpha}\hat{\beta}} - \frac{2}{3}\bar{S}_{ij}|\zeta^{ij} \right), \quad (3.20)$$

where $\mathcal{E} = \text{Ber}(\mathcal{E}_{\hat{m}}^{\hat{a}})$ is the chiral density.

Relation (3.20) can be used to reduce an integral over the full superspace to that over the chiral subspace. Consider the functional

$$\int d^4x d^4\theta d^4\bar{\theta} E U = \int d^4x d^4\theta d^4\zeta E(\tilde{z}, \zeta) U(\tilde{z}, \zeta), \quad (3.21)$$

where $U(z)$ is a scalar and isoscalar superfield, and $\tilde{z}^{\hat{m}} = (x^m, \theta_i^\mu)$ the variables parametrizing the chiral subspace. In the normal coordinates, one represents U by its covariant Taylor

expansion in ζ , eq. (2.52), then evaluates the product EU , and finally performs the integration over $d^4\zeta$. The result is as follows:

$$\int d^4x d^4\theta d^4\bar{\theta} EU = \int d^4x d^4\theta \mathcal{E} \bar{\Delta}U | . \quad (3.22)$$

Here $\bar{\Delta}$ denotes the following fourth-order operator:

$$\begin{aligned} \bar{\Delta} &= \frac{1}{96} \left((\bar{\mathcal{D}}^{ij} + 16\bar{S}^{ij})\bar{\mathcal{D}}_{ij} - (\bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} - 16\bar{Y}^{\dot{\alpha}\dot{\beta}})\bar{\mathcal{D}}_{\dot{\alpha}\dot{\beta}} \right) \\ &= \frac{1}{96} \left(\bar{\mathcal{D}}_{ij}(\bar{\mathcal{D}}^{ij} + 16\bar{S}^{ij}) - \bar{\mathcal{D}}_{\dot{\alpha}\dot{\beta}}(\bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} - 16\bar{Y}^{\dot{\alpha}\dot{\beta}}) \right), \end{aligned} \quad (3.23)$$

where we have defined

$$\bar{\mathcal{D}}^{\dot{\alpha}\dot{\beta}} := \bar{\mathcal{D}}_k^{(\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\beta})k}, \quad \bar{\mathcal{D}}_{ij} := \bar{\mathcal{D}}_{\dot{\gamma}(i} \bar{\mathcal{D}}_{j)}^{\dot{\gamma}} . \quad (3.24)$$

The operator $\bar{\Delta}$ is the $\mathcal{N} = 2$ covariantly chiral projector [18]. Its fundamental property is that $\bar{\Delta}U$ is covariantly chiral, for any scalar and isoscalar superfield $U(z)$,

$$\bar{\mathcal{D}}_{\dot{i}}^{\dot{\alpha}} \bar{\Delta}U = 0 . \quad (3.25)$$

In section 5, we obtain a different representation for the chiral projector.

4 Density formula in $\mathcal{N} = 2$ supergravity

In this section, the supergravity action (1.2) is reduced to components using the principle of projective invariance. We start by elaborating some auxiliary tools.

4.1 Relating the superspace and the space-time covariant derivatives

For any superfield $U(z)$ we define its projection $U|$ to be the lowest component in the expansion of $U(x, \theta, \bar{\theta})$ with respect to θ s and $\bar{\theta}$ s,

$$U(z)| := U(x, \theta, \bar{\theta})|_{\theta=\bar{\theta}=0} . \quad (4.1)$$

One can similarly define the projection of the covariant derivatives:

$$\mathcal{D}_A| := E_A^M(z)|\partial_M + \frac{1}{2}\Omega_A^{bc}(z)|M_{bc} + \Phi_A^{kl}(z)|J_{kl} . \quad (4.2)$$

More generally, given a gauge covariant operator of the form $\mathcal{D}_{A_1} \dots \mathcal{D}_{A_n}$, its projection $(\mathcal{D}_{A_1} \dots \mathcal{D}_{A_n})|$ is defined as

$$\left((\mathcal{D}_{A_1} \dots \mathcal{D}_{A_n})|U \right)| := (\mathcal{D}_{A_1} \dots \mathcal{D}_{A_n}U)| , \quad (4.3)$$

with U an arbitrary tensor superfield. The reader should keep in mind that the projection operation defined above differs from that used in section 3.

In the case of the vector covariant derivatives, \mathcal{D}_a , their projection can be represented in the form:

$$\mathcal{D}_a| = \nabla_a + \Psi_{a\dot{k}}^{\dot{\gamma}}(x)\mathcal{D}_{\dot{\gamma}}^{\dot{k}}| + \bar{\Psi}_{a\dot{\gamma}}^{\dot{k}}(x)\bar{\mathcal{D}}_{\dot{k}}^{\dot{\gamma}}| + \phi_a^{kl}(x)J_{kl} , \quad (4.4)$$

with ∇_a a space-time covariant derivative,

$$\nabla_a = e_a + \omega_a, \quad e_a = e_a^m(x)\partial_m, \quad \omega_a = \frac{1}{2}\omega_a^{bc}(x)M_{bc}. \quad (4.5)$$

Here we have introduced several component gauge fields defined as follows:

$$E_a^m(z)| = e_a^m(x) + \Psi_{a_k}^\gamma(x)E_\gamma^{km}(z)| + \bar{\Psi}_{a\dot{\gamma}}^k(x)E_{\dot{\gamma}}^{\dot{m}}(z)|, \quad (4.6a)$$

$$E_{a_r}^\mu(z)| = \Psi_{a_k}^\gamma(x)E_{\gamma r}^{k\mu}(z)| + \bar{\Psi}_{a\dot{\gamma}}^k(x)E_{\dot{\gamma} r}^{\dot{\mu}}(z)|, \quad (4.6b)$$

$$E_{a\dot{\mu}}^r(z)| = \Psi_{a_k}^\gamma(x)E_{\gamma\dot{\mu}}^{kr}(z)| + \bar{\Psi}_{a\dot{\gamma}}^k(x)E_{\dot{\gamma}}^{\dot{r}}(z)|, \quad (4.6c)$$

$$\Omega_a^{bc}(z)| = \omega_a^{bc}(x) + \Psi_{a_k}^\gamma(x)\Omega_\gamma^{kbc}(z)| + \bar{\Psi}_{a\dot{\gamma}}^k(x)\Omega_{\dot{\gamma}}^{\dot{bc}}(z)|, \quad (4.6d)$$

$$\Phi_a^{kl}(z)| = \phi_a^{kl}(x) + \Psi_{a_j}^\beta(x)\Omega_\beta^{jkl}(z)| + \bar{\Psi}_{a\dot{\beta}}^j(x)\Omega_{\dot{\beta}}^{\dot{kl}}(z)|. \quad (4.6e)$$

These include the inverse vielbein e_a^m , the Lorentz connection ω_a^{bc} and the SU(2)-connection ϕ_a^{kl} , as well as the gravitino fields $\Psi_{a_k}^\gamma$ and $\bar{\Psi}_{a\dot{\gamma}}^k$.

It is worth noting that if one chooses an $\mathcal{N} = 2$ analogue of Wess-Zumino gauge [13] defined as

$$\mathcal{D}_\alpha^i| = \frac{\partial}{\partial\theta_\alpha^i}, \quad \bar{\mathcal{D}}_{\dot{\alpha}}^i| = \frac{\partial}{\partial\theta_{\dot{\alpha}}^i}, \quad (4.7)$$

then the relations (4.6a)–(4.6e) considerably simplify and take the form:

$$E_a^m(z)| = e_a^m(x), \quad E_{a_k}^\gamma(z)| = \Psi_{a_k}^\gamma(x), \quad E_{a\dot{\gamma}}^k(z)| = \bar{\Psi}_{a\dot{\gamma}}^k(x), \quad (4.8a)$$

$$\Omega_a^{bc}(z)| = \omega_a^{bc}(x), \quad \Phi_a^{kl}(z)| = \phi_a^{kl}(x). \quad (4.8b)$$

The space-time covariant derivatives obey the commutation relations

$$[\nabla_a, \nabla_b] = \mathcal{T}_{ab}^c(x)\nabla_c + \frac{1}{2}\mathcal{R}_{ab}^{cd}(x)M_{cd}. \quad (4.9)$$

Here the torsion tensor determines the rule for integration by parts:

$$\int d^4x e \nabla_a v^a = \int d^4x e v^a \mathcal{T}_{ab}^b, \quad e^{-1} := \det(e_a^m), \quad (4.10)$$

with v^a an arbitrary vector field.

The space-time torsion \mathcal{T}_{ab}^c and curvature \mathcal{R}_{ab}^{cd} can be related to those appearing in the superspace (anti-)commutation relations (A.9a)–(A.9e). Using the definition (4.4) and eqs. (A.9a)–(A.9e), one can evaluate the projection of the commutator $[\mathcal{D}_a, \mathcal{D}_b]$ to be

$$\begin{aligned} [\mathcal{D}_a, \mathcal{D}_b]| &= \mathcal{T}_{ab}^c \nabla_c - 4i \Psi_{[a_k}^\gamma \bar{\Psi}_{b]\dot{\delta}}^k \nabla_{\dot{\gamma}}^{\dot{\delta}} + \frac{1}{2} \mathcal{R}_{ab}^{cd} M_{cd} - \Psi_{[a_k}^\gamma R_{b]\gamma}^{kcd} | M_{cd} \\ &+ \frac{1}{2} \Psi_{[a_k}^\gamma \bar{\Psi}_{b]l}^\delta R_{\gamma\delta}^{klcd} | M_{cd} + \frac{1}{2} \bar{\Psi}_{[a\dot{\gamma}}^k \bar{\Psi}_{b]\dot{\delta}}^l R_{\dot{\gamma}\dot{\delta}}^{klcd} | M_{cd} + \Psi_{[a_k}^\gamma \bar{\Psi}_{b]l}^\delta R_{\gamma l}^{k\dot{\delta}cd} | M_{cd} + 2(\nabla_{[a} \Psi_{b]k}^\gamma) \mathcal{D}_\gamma^k \\ &- 2\Psi_{[a_i}^\alpha T_{b]\alpha k}^i \mathcal{D}_\gamma^k | - 2\bar{\Psi}_{[a\dot{\alpha}}^i T_{b]i k}^{\dot{\alpha}\gamma} \mathcal{D}_\gamma^k | - 2\phi_{[ak}^l \bar{\Psi}_{b]l}^\gamma \mathcal{D}_\gamma^k | - 4i \Psi_{[a_k}^\gamma \bar{\Psi}_{b]\dot{\delta}}^k \Psi_{\dot{\gamma}}^{\dot{\delta}\gamma} \mathcal{D}_\gamma^k | \\ &+ 2(\nabla_{[a} \bar{\Psi}_{b]\dot{\gamma}}^k) \bar{\mathcal{D}}_{\dot{\delta}}^{\dot{\gamma}} | - 2\Psi_{[a_i}^\alpha T_{b]i k}^{\alpha\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\delta}}^{\dot{\gamma}} | - 2\bar{\Psi}_{[a\dot{\gamma}}^k T_{b]k\dot{\delta}}^{\dot{\gamma}\alpha} \bar{\mathcal{D}}_{\dot{\delta}}^{\dot{\alpha}} | + 2\phi_{[a}^k \bar{\Psi}_{b]\dot{\gamma}}^l \bar{\mathcal{D}}_{\dot{\delta}}^{\dot{\gamma}} | \\ &- 4i \Psi_{[a_k}^\gamma \bar{\Psi}_{b]\dot{\delta}}^k \bar{\Psi}_{\dot{\gamma}}^{\dot{\delta}\gamma} \bar{\mathcal{D}}_{\dot{\delta}}^{\dot{\gamma}} | + 2(\nabla_{[a} \phi_{b]}^{kl}) J_{kl} - 2\Psi_{[a_j}^\gamma R_{b]\gamma}^{jkl} | J_{kl} - 2\bar{\Psi}_{[a\dot{\gamma}}^j R_{b]\dot{\gamma}}^{\dot{\gamma}kl} | J_{kl} \end{aligned}$$

$$\begin{aligned}
 & + \Psi_{[ai}{}^\gamma \bar{\Psi}_{b]j}{}^\delta R_{\gamma\delta}{}^{ijkl} |J_{kl} + \bar{\Psi}_{[a\dot{\gamma}}{}^i \bar{\Psi}_{b]j}{}^{\dot{\delta}} R_{i\dot{j}}{}^{\dot{\delta}kl} |J_{kl} + 2\Psi_{[ai}{}^\gamma \bar{\Psi}_{b]j}{}^\delta R_{\gamma j}{}^{\delta kl} |J_{kl} + 2\phi_{[a}{}^k {}_j\phi_b]{}^{jl} J_{kl} \\
 & - 4i\Psi_{[aj}{}^\gamma \bar{\Psi}_{b]j}{}^\delta \phi_{\gamma}{}^{\delta kl} J_{kl} .
 \end{aligned} \tag{4.11}$$

On the other hand, the commutator $[\mathcal{D}_a, \mathcal{D}_b]$ can be evaluated using eqs. (A.9a)–(A.9e). Comparing the similar structures on both sides gives a number of important relations including the following:

$$T_{ab}{}^c = 4i\Psi_{[ak}{}^\gamma \bar{\Psi}_{b]j}{}^k (\sigma^c)_{\gamma}{}^{\dot{\delta}} , \tag{4.12a}$$

$$(\nabla_{[a} \Psi_{b]k}{}^\gamma) = \frac{1}{2} T_{abk}{}^\gamma + \Psi_{[ai}{}^\alpha T_{b]k}{}^{i\gamma} + \bar{\Psi}_{[a\dot{\alpha}}{}^i T_{b]i}{}^{\dot{\alpha}\gamma} + \phi_{[ak}{}^l \bar{\Psi}_{b]l}{}^\gamma + 2i\Psi_{[al}{}^\delta \bar{\Psi}_{b]j}{}^l \Psi_{\delta}{}^{\dot{\delta}\gamma} , \tag{4.12b}$$

$$(\nabla_{[a} \bar{\Psi}_{b]j}{}^k) = \frac{1}{2} T_{abj}{}^k + \Psi_{[ai}{}^\alpha T_{b]j}{}^{i\dot{\alpha}k} + \bar{\Psi}_{[a\dot{\alpha}}{}^k T_{b]i}{}^{\dot{\alpha}k} - \phi_{[a}{}^k {}_l \bar{\Psi}_{b]j}{}^l + 2i\Psi_{[al}{}^\delta \bar{\Psi}_{b]j}{}^l \bar{\Psi}_{\delta}{}^{\dot{\delta}k} , \tag{4.12c}$$

$$\begin{aligned}
 \mathcal{R}_{ab}{}^{cd} & = R_{ab}{}^{cd} + 2\Psi_{[ak}{}^\gamma R_{b]j}{}^{kcd} + 2\bar{\Psi}_{[a\dot{\gamma}}{}^k R_{b]j}{}^{\dot{\gamma}cd} - \Psi_{[ak}{}^\gamma \Psi_{b]l}{}^\delta R_{\gamma\delta}{}^{klcd} \\
 & - \bar{\Psi}_{[a\dot{\gamma}}{}^k \bar{\Psi}_{b]j}{}^l R_{kl}{}^{\dot{\delta}cd} - 2\Psi_{[ak}{}^\gamma \bar{\Psi}_{b]j}{}^l R_{\gamma l}{}^{\dot{\delta}cd} ,
 \end{aligned} \tag{4.12d}$$

$$\begin{aligned}
 (\nabla_{[a} \phi_b]{}^{kl}) & = \frac{1}{2} R_{ab}{}^{kl} + \Psi_{[aj}{}^\gamma R_{b]j}{}^{kl} + \bar{\Psi}_{[a\dot{j}}{}^j R_{b]j}{}^{\dot{\gamma}kl} - \frac{1}{2} \Psi_{[ai}{}^\gamma \Psi_{b]j}{}^\delta R_{\gamma\delta}{}^{ijkl} - \frac{1}{2} \bar{\Psi}_{[a\dot{i}}{}^i \bar{\Psi}_{b]j}{}^{\dot{j}} R_{i\dot{j}}{}^{\dot{\delta}kl} \\
 & - \Psi_{[ai}{}^\gamma \bar{\Psi}_{b]j}{}^{\dot{\delta}} R_{\gamma j}{}^{\delta kl} - \phi_{[a}{}^k {}_j\phi_b]{}^{jl} + 2i\Psi_{[aj}{}^\gamma \bar{\Psi}_{b]j}{}^{\dot{\delta}} \phi_{\gamma}{}^{\delta kl} .
 \end{aligned} \tag{4.12e}$$

In what follows, we will often use eq. (4.12a), (4.12b) and (4.12c).

4.2 The component action

We turn to demonstrating that the component reduction of action (1.2) is

$$\begin{aligned}
 S & = \oint_C d\mu^{(-2,-4)} \int d^4x e \left[\frac{1}{16} (\mathcal{D}^-)^2 (\bar{\mathcal{D}}^-)^2 + \frac{3}{4} S^{--} (\bar{\mathcal{D}}^-)^2 + \frac{3}{4} \bar{S}^{--} (\mathcal{D}^-)^2 + 9S^{--} \bar{S}^{--} \right. \\
 & + \frac{i}{4} \Psi^{\alpha\dot{\alpha}-} (\mathcal{D}^-)^2 \bar{\mathcal{D}}_{\dot{\alpha}}^- + \frac{i}{4} \bar{\Psi}^{\alpha\dot{\alpha}-} (\bar{\mathcal{D}}^-)^2 \mathcal{D}_{\alpha}^- - \phi^{\alpha\dot{\alpha}-} \mathcal{D}_{\alpha}^- \bar{\mathcal{D}}_{\dot{\alpha}}^- \\
 & + (\sigma^{ab})^{\alpha\beta} \Psi_{a\alpha}{}^- \left(\Psi_{b\beta}{}^- (\mathcal{D}^-)^2 + 2\bar{\Psi}_b{}^{\dot{\beta}-} \mathcal{D}_{\beta}^- \bar{\mathcal{D}}_{\dot{\beta}}^- \right) + (\tilde{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{\Psi}_{a\dot{\alpha}}{}^- \left(\bar{\Psi}_{b\dot{\beta}}{}^- (\bar{\mathcal{D}}^-)^2 + 2\Psi_b{}^{\beta-} \mathcal{D}_{\beta}^- \bar{\mathcal{D}}_{\dot{\beta}}^- \right) \\
 & + 3i \left(\bar{\Psi}^{\alpha\dot{\alpha}-} \bar{S}^{--} \mathcal{D}_{\alpha}^- + \Psi^{\alpha\dot{\alpha}-} S^{--} \bar{\mathcal{D}}_{\dot{\alpha}}^- \right) - 4\phi_a{}^{--} \left((\sigma^{ab})^{\beta\gamma} \Psi_{b\beta}{}^- \mathcal{D}_{\gamma}^- - (\tilde{\sigma}^{ab})^{\dot{\beta}\dot{\gamma}} \bar{\Psi}_{b\dot{\beta}}{}^- \bar{\mathcal{D}}_{\dot{\gamma}}^- \right) \\
 & + 4\varepsilon^{abcd} (\sigma_d)_{\alpha\dot{\beta}} \Psi_a{}^{\alpha-} \bar{\Psi}_b{}^{\dot{\beta}-} \left(\Psi_c{}^{\gamma-} \mathcal{D}_{\gamma}^- + \bar{\Psi}_c{}^{\dot{\gamma}-} \bar{\mathcal{D}}_{\dot{\gamma}}^- \right) - 12\varepsilon^{abcd} (\sigma_d)_{\alpha\dot{\beta}} \Psi_a{}^{\alpha-} \bar{\Psi}_b{}^{\dot{\beta}-} \phi_c{}^{--} \\
 & \left. + 12(\sigma^{ab})^{\alpha\beta} \Psi_{a\alpha}{}^- \Psi_{b\beta}{}^- S^{--} + 12(\tilde{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{\Psi}_{a\dot{\alpha}}{}^- \bar{\Psi}_{b\dot{\beta}}{}^- \bar{S}^{--} \right] \mathcal{L}^{++}(z, u^+) ,
 \end{aligned} \tag{4.13}$$

where

$$S^{\pm\pm} := u_i^{\pm} u_j^{\pm} S^{ij} , \quad \Psi_{a\alpha}^{\pm} := u_i^{\pm} \Psi_{a\alpha}^i , \quad \phi_a{}^{\pm\pm} := u_i^{\pm} u_j^{\pm} \phi_a{}^{ij} , \tag{4.14}$$

and similarly for $\bar{S}^{\pm\pm}$ and $\bar{\Psi}_{a\dot{\alpha}}^{\pm}$. The spinor derivatives \mathcal{D}_{α}^- and $\bar{\mathcal{D}}_{\dot{\alpha}}^-$ are obtained from \mathcal{D}_{α}^+ and $\bar{\mathcal{D}}_{\dot{\alpha}}^+$ defined in (1.1) by replacing $u_i^+ \rightarrow u_i^-$. The contour integration measure in (4.13) is defined as follows:

$$d\mu^{(-2,-4)} \equiv -\frac{1}{2\pi} \frac{u_i^+ du^{+i}}{(u^+ u^-)^4} = -\frac{1}{2\pi} \frac{(\dot{u}^+ u^+)}{(u^+ u^-)^4} dt , \tag{4.15}$$

with t an evolution parameter along the contour C , and $\dot{f} := df(t)/dt$ the time derivative of a function $f(t)$. Here u_i^- is a constant isotwistor subject only to the restriction that u_i^- and

$u_i^+(t)$ are linearly independent at each point of the closed contour C , that is $(u^+u^-) \neq 0$. The remainder of this section is devoted to the derivation of (4.13).

In what follows, we often change bases in the space of isotensors by the rule $A^i \rightarrow A^\pm := A^i u_i^\pm$ using the completeness relation

$$(u^+u^-) \delta_j^i = u^{+i} u_j^- - u^{-i} u_j^+ . \tag{4.16}$$

We also find it helpful to introduce a notational convention that differs slightly from that used in [1–3]. Specifically, $F^{(p,q)}(u^+, u^-)$ denotes a homogeneous function of u^+ s and u^- s, with integers p and q being the corresponding degrees of homogeneity with respect to u^+ s and u^- s, that is: $F^{(p,q)}(c u^+, u^-) = c^p F^{(p,q)}(u^+, u^-)$ and $F^{(p,q)}(u^+, c u^-) = c^q F^{(p,q)}(u^+, u^-)$, where $c \in \mathbb{C} \setminus \{0\}$. This convention is reflected in the definition (4.15). In the case of a homogeneous function of u^+ s only, we use the simplified notation: $F^{(n)}(u^+) \equiv F^{(n,0)}(u^+)$; if $n > 0$, we can also write $F^{(n)} \equiv F^{+\dots+}$, where the number of + superscripts is equal to n . In the case of a homogeneous function of u^- s only, we often use the simplified notation $F^{-\dots-}(u^-) \equiv F^{(0,m)}(u^-)$ with $m > 0$, where the number of – superscripts is equal to m .

A few words are in order regarding our strategy of deriving (4.13). It is clear that the component Lagrangian corresponding to the action (1.2) should be a combination of terms with four and less spinor covariant derivatives acting on \mathcal{L}^{++} . In the complete set of spinor covariant derivatives, \mathcal{D}_α^i and $\bar{\mathcal{D}}_{\dot{\alpha}}^i$, these derivatives should be linearly independent from the operators \mathcal{D}_α^+ and $\bar{\mathcal{D}}_{\dot{\alpha}}^+$ which annihilate \mathcal{L}^{++} . A natural way to define such a subset of spinor covariant derivatives is to pick an isotwistor u_i^- such that $(u^+u^-) \neq 0$. Then the operators \mathcal{D}_α^- and $\bar{\mathcal{D}}_{\dot{\alpha}}^-$ clearly satisfy the required criterion. In other words, in order to construct the component action one is forced to introduce an external isotwistor u_i^- which does not show up in the original action (1.2).¹⁴ The latter involves only the isotwistor u_i^+ , and is invariant under arbitrary re-scalings

$$u_i^+(t) \rightarrow c(t) u_i^+(t), \quad c(t) \neq 0, \tag{4.17}$$

along the integration contour. Therefore, the component action should be invariant under arbitrary projective transformations (1.6). Indeed, the invariance under infinitesimal transformations of the form

$$u_i^- \rightarrow u_i^- + \delta u_i^-, \quad \delta u_i^- = \alpha(t) u_i^- + \beta(t) u_i^+(t), \tag{4.18}$$

implies independence of the action from the choice of u_i^- . Since both u_i^- and δu_i^- are required to be time-independent, the transformation parameters should obey the equations:

$$\dot{\alpha} = \beta \frac{(\dot{u}^+ u^+)}{(u^+ u^-)}, \quad \dot{\beta} = -\beta \frac{(\dot{u}^+ u^-)}{(u^+ u^-)}. \tag{4.19}$$

Setting $\beta = 0$ in (4.18) gives a scale transformation, $\delta u_i^- = \alpha u_i^-$. Therefore, the component action must be invariant under arbitrary rigid re-scalings of u_i^- . If the component

¹⁴This is similar to the Faddeev-Popov quantization of Yang-Mills theories. In order to develop a path-integral representation for the vacuum amplitude $\langle out|in \rangle$, one has to introduce a gauge fixing condition. However, the amplitude $\langle out|in \rangle$ must be independent of the gauge condition introduced.

Lagrangian density is chosen to be homogeneous in u_i^- of degree zero, then the invariance under rigid re-scalings of u_i^- clearly extends to that under the time-dependent α -transformations in (4.18). It turns out that a nontrivial piece of information is provided by requiring the action to be invariant under the β -transformations in (4.18).

On general grounds, it is not difficult to fix a four-derivative term in the component Lagrangian corresponding to the action (1.2). We have

$$S = S_0 + \dots, \quad S_0 = \frac{1}{16} \oint d\mu^{(-2,-4)} \int d^4x e (\mathcal{D}^-)^2 (\bar{\mathcal{D}}^-)^2 \mathcal{L}^{++}(z, u^+) \Big|, \quad (4.20)$$

where the dots denote all the terms with at the most three spinor derivatives. The functional S_0 is obviously invariant under the local re-scalings of u_i^+ , eq. (4.17), and also under the α -transformations in (4.18). It turns out, however, that S_0 is not invariant under the β -transformation in (4.18). To cancel out the β -variation of S_0 , it is necessary to add to S_0 some terms with three and less spinor derivatives acting on \mathcal{L}^{++} . The latter produce new non-vanishing contributions of lower order under the the β -transformation in (4.18). As a result, we end up with a well-defined iterative procedure to restore a projective invariant action. Conceptually, our approach below is quite simple.

Before proceeding with the computation, it is useful to collect some auxiliary results and make a technical comment. Since the superfield Lagrangian $\mathcal{L}^{++}(z, u^+)$ is a weight-two projective supermultiplet, it holds that

$$J_{kl} \mathcal{L}^{++} = -\frac{1}{(u^+ u^-)} \left(u_{(k}^+ u_{l)}^+ D^{(-1,1)} - 2u_{(k}^+ u_{l)}^- \right) \mathcal{L}^{++}, \quad (4.21a)$$

$$\frac{d}{dt} \mathcal{L}^{++} = 2 \frac{(\dot{u}^+ u^-)}{(u^+ u^-)} \mathcal{L}^{++} - \frac{(\dot{u}^+ u^+)}{(u^+ u^-)} D^{(-1,1)} \mathcal{L}^{++}, \quad (4.21b)$$

$$(\dot{u}^+ u^+) J_{kl} \mathcal{L}^{++} = u_{(k}^+ u_{l)}^+ \frac{d}{dt} \mathcal{L}^{++} - 2 \frac{(\dot{u}^+ u^-)}{(u^+ u^-)} u_{(k}^+ u_{l)}^+ \mathcal{L}^{++} + 2 \frac{(\dot{u}^+ u^+)}{(u^+ u^-)} u_{(k}^+ u_{l)}^- \mathcal{L}^{++}, \quad (4.21c)$$

with J_{kl} the SU(2) generators. Here the operator $D^{(-1,1)}$ is defined in (B.3). Consider now any operator \mathcal{O}^{--} , which is independent of u^+ , $\partial \mathcal{O}^{--} / \partial u^{+i} = 0$, and is homogeneous in the variables u_i^- of degree +2. Using equations (4.21a)–(4.21c), one gets

$$\frac{(\dot{u}^+ u^+)}{(u^+ u^-)^4} \mathcal{O}^{--} J^{--} \mathcal{L}^{++} = \frac{d}{dt} \left[\frac{\mathcal{O}^{--}}{(u^+ u^-)^2} \mathcal{L}^{++} \right]. \quad (4.22)$$

This implies the following relation:

$$\oint d\mu^{(-2,-4)} \mathcal{O}^{--} J^{--} \mathcal{L}^{++} = 0. \quad (4.23)$$

Due to the identities

$$[J_{kl}, \mathcal{D}_\alpha^\pm] = \frac{u_{(k}^\pm u_{l)}^\mp}{(u^+ u^-)} \mathcal{D}_\alpha^\pm - \frac{u_{(k}^\pm u_{l)}^\pm}{(u^+ u^-)} \mathcal{D}_\alpha^\mp, \quad [J_{kl}, \bar{\mathcal{D}}_{\dot{\alpha}}^\pm] = \frac{u_{(k}^\pm u_{l)}^\mp}{(u^+ u^-)} \bar{\mathcal{D}}_{\dot{\alpha}}^\pm - \frac{u_{(k}^\pm u_{l)}^\pm}{(u^+ u^-)} \bar{\mathcal{D}}_{\dot{\alpha}}^\mp, \quad (4.24)$$

$$\{\mathcal{D}_\alpha^-, \bar{\mathcal{D}}_{\dot{\alpha}}^-\} = 8G_{\alpha\dot{\alpha}} J^{--}, \quad [J^{--}, \mathcal{D}_\alpha^-] = [J^{--}, \bar{\mathcal{D}}_{\dot{\alpha}}^-] = 0, \quad (4.25)$$

we also obtain

$$\begin{aligned} \oint d\mu^{(-2,-4)} (\mathcal{D}^-)^2 (\bar{\mathcal{D}}^-)^2 \mathcal{L}^{++} &= \oint d\mu^{(-2,-4)} \mathcal{D}^{\alpha-} (\bar{\mathcal{D}}^-)^2 \mathcal{D}_\alpha^- \mathcal{L}^{++} \\ &= \oint d\mu^{(-2,-4)} (\bar{\mathcal{D}}^-)^2 (\mathcal{D}^-)^2 \mathcal{L}^{++} = \oint d\mu^{(-2,-4)} \bar{\mathcal{D}}_{\dot{\alpha}}^- (\mathcal{D}^-)^2 \bar{\mathcal{D}}^{\dot{\alpha}-} \mathcal{L}^{++}. \end{aligned} \quad (4.26)$$

These identities justify the fact that S_0 is unambiguously defined.

Using equations (4.19) and (4.21a)–(4.21c), one can also prove (compare with the similar observation in the 5D case [2]) the following result: for any operator $\mathcal{O}^{(1,3)kl}$, which is an homogenous function of degrees 1 and 3 in u_i^+ and u_i^- , respectively, it holds that

$$\begin{aligned} \oint d\mu^{(-2,-4)} \beta \mathcal{O}^{(1,3)kl} J_{kl} \mathcal{L}^{++} \\ = \oint d\mu^{(-2,-4)} \frac{\beta}{(u^+ u^-)} \left\{ 4 \mathcal{O}^{(1,3)++} \mathcal{L}^{++} + u_k^+ u_l^+ \left(D^{(-1,1)} \mathcal{O}^{(1,3)kl} \right) \mathcal{L}^{++} \right\}. \end{aligned} \quad (4.27)$$

This identity will often be used in what follows.

Let us consider the variation of S_0 , eq. (4.20), under the infinitesimal projective transformation (4.18). Since $\mathcal{D}_\alpha^+ \mathcal{L}^{++} = \bar{\mathcal{D}}_{\dot{\alpha}}^+ \mathcal{L}^{++} = 0$, we obtain

$$\begin{aligned} \delta S_0 = \frac{1}{16} \oint d\mu^{(-2,-4)} \beta \int d^4 x e \left[\{ \mathcal{D}^{\alpha+}, \mathcal{D}_\alpha^- \bar{\mathcal{D}}_{\dot{\alpha}}^- \bar{\mathcal{D}}^{\dot{\alpha}-} \} + \mathcal{D}^{\alpha-} [\mathcal{D}_\alpha^+, \bar{\mathcal{D}}_{\dot{\alpha}}^- \bar{\mathcal{D}}^{\dot{\alpha}-}] \right. \\ \left. + \mathcal{D}^{\alpha-} \mathcal{D}_\alpha^- \{ \bar{\mathcal{D}}_{\dot{\alpha}}^+, \bar{\mathcal{D}}^{\dot{\alpha}-} \} \right] \mathcal{L}^{++}, \end{aligned} \quad (4.28)$$

which is equivalent to

$$\begin{aligned} \delta S_0 = \frac{1}{16} \oint d\mu^{(-2,-4)} \beta \int d^4 x e \left[\{ \mathcal{D}^{\alpha+}, \mathcal{D}_\alpha^- \} \bar{\mathcal{D}}_{\dot{\alpha}}^- \bar{\mathcal{D}}^{\dot{\alpha}-} + 4 \{ \mathcal{D}_\alpha^+, \bar{\mathcal{D}}_{\dot{\alpha}}^- \} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} \right. \\ \left. - 4 \{ \{ \mathcal{D}_\alpha^-, \bar{\mathcal{D}}_{\dot{\alpha}}^- \}, \mathcal{D}^{\alpha+} \} \bar{\mathcal{D}}^{\dot{\alpha}-} - 4 \{ \{ \mathcal{D}^{\alpha+}, \mathcal{D}_\alpha^- \}, \bar{\mathcal{D}}_{\dot{\alpha}}^- \} \bar{\mathcal{D}}^{\dot{\alpha}-} - 2 \mathcal{D}^{\alpha-} \{ \{ \mathcal{D}_\alpha^+, \bar{\mathcal{D}}_{\dot{\alpha}}^- \}, \bar{\mathcal{D}}^{\dot{\alpha}-} \} \right. \\ \left. + \mathcal{D}^{\alpha-} \mathcal{D}_\alpha^- \{ \bar{\mathcal{D}}_{\dot{\alpha}}^+, \bar{\mathcal{D}}^{\dot{\alpha}-} \} \right] \mathcal{L}^{++}. \end{aligned} \quad (4.29)$$

Here the (anti)commutators can be evaluated by making use of the algebra (A.9a)–(A.9e). As a next step, we systematically move the Lorentz and SU(2) generators to the right and then use the identity $M_{ab} \mathcal{L}^{++} = 0$ and eq. (4.27). If in this process some spinor covariant derivatives \mathcal{D}_α^+ or $\bar{\mathcal{D}}_{\dot{\alpha}}^+$ are produced, we push them to the right until they hit \mathcal{L}^{++} , and the latter contribution vanishes due to $\mathcal{D}_\alpha^+ \mathcal{L}^{++} = \bar{\mathcal{D}}_{\dot{\alpha}}^+ \mathcal{L}^{++} = 0$. We then find

$$\begin{aligned} \delta S_0 = \frac{1}{16} \oint d\mu^{(-2,-4)} \beta \int d^4 x e \left[-8i(u^+ u^-) \mathcal{D}_{\alpha\dot{\alpha}} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} - 24S^{+-} \bar{\mathcal{D}}_{\dot{\alpha}}^- \bar{\mathcal{D}}^{\dot{\alpha}-} - 24\bar{S}^{+-} \mathcal{D}^{\alpha-} \mathcal{D}_\alpha^- \right. \\ \left. - 16(u^+ u^-) (\bar{\mathcal{D}}_{\dot{\alpha}}^- \bar{W}^{\dot{\alpha}\dot{\delta}}) \bar{\mathcal{D}}_{\dot{\delta}}^- - 48(\bar{\mathcal{D}}_{\dot{\alpha}}^- S^{+-}) \bar{\mathcal{D}}^{\dot{\alpha}-} - 56(\mathcal{D}^{\beta-} \bar{S}^{+-}) \mathcal{D}_\beta^- \right. \\ \left. + 8(u^+ u^-) (\mathcal{D}^{\beta-} W_{\beta\gamma}) \mathcal{D}^{\gamma-} + 16(u^+ u^-) (\bar{\mathcal{D}}^{\dot{\alpha}-} G_{\alpha\dot{\alpha}}) \mathcal{D}^{\alpha-} \right. \\ \left. - 192S^{--} \bar{S}^{+-} - 32(\mathcal{D}^{\beta-} \mathcal{D}_\beta^- \bar{S}^{+-}) - 16(u^+ u^-) (\mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} G_{\alpha\dot{\alpha}}) \right] \mathcal{L}^{++}. \end{aligned} \quad (4.30)$$

This expression can be simplified if one notices that the Bianchi identities (A.11)–(A.14) imply

$$\mathcal{D}_\alpha^+ S^{--} = -2\mathcal{D}_\alpha^- S^{+-}, \quad \mathcal{D}_{\alpha l} S^{-l} = \frac{3}{(u^+ u^-)} \mathcal{D}_\alpha^- S^{+-}, \quad (4.31a)$$

$$\bar{\mathcal{D}}^{\dot{\alpha}-} G_{\alpha\dot{\alpha}} = \frac{1}{4(u^+u^-)} \mathcal{D}_\alpha^+ \bar{S}^{--} + \frac{1}{2} \mathcal{D}^{\gamma-} W_{\alpha\gamma}, \quad (4.31b)$$

$$\mathcal{D}^{\alpha-} \mathcal{D}^{\beta-} W_{\alpha\beta} = 0, \quad \mathcal{D}^{\alpha-} \mathcal{D}_\alpha^- \bar{S}^{+-} = 4S^{+-} \bar{S}^{--} - 4S^{--} \bar{S}^{+-}, \quad (4.31c)$$

$$\mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} G_{\alpha\dot{\alpha}} = -\frac{2}{(u^+u^-)} S^{+-} \bar{S}^{--} + \frac{2}{(u^+u^-)} S^{--} \bar{S}^{+-}, \quad (4.31d)$$

along with complex conjugate relations. We then end up with the following variation:

$$\begin{aligned} \delta S_0 = \oint d\mu^{(-2,-4)} \beta \int d^4x e \left[-\frac{i}{2} (u^+u^-) \mathcal{D}_{\alpha\dot{\alpha}} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} - \frac{3}{2} S^{+-} \bar{\mathcal{D}}_\alpha^- \bar{\mathcal{D}}^{\dot{\alpha}-} - \frac{3}{2} \bar{S}^{+-} \mathcal{D}^{\alpha-} \mathcal{D}_\alpha^- \right. \\ \left. - 3(\bar{\mathcal{D}}_\alpha^- S^{+-}) \bar{\mathcal{D}}^{\dot{\alpha}-} - 3(\mathcal{D}^{\alpha-} \bar{S}^{+-}) \mathcal{D}_\alpha^- - (u^+u^-) (\bar{\mathcal{D}}_\alpha^- \bar{W}^{\dot{\alpha}\dot{\delta}}) \bar{\mathcal{D}}_{\dot{\delta}}^- + (u^+u^-) (\mathcal{D}^{\alpha-} W_{\alpha\beta}) \mathcal{D}^{\beta-} \right. \\ \left. - 6S^{--} \bar{S}^{+-} - 6S^{+-} \bar{S}^{--} \right] \mathcal{L}^{++} \Big| . \end{aligned} \quad (4.32)$$

To cancel out the terms with two derivatives, we add to S_0 the following structure:

$$S_1 = \oint d\mu^{(-2,-4)} \int d^4x e \left[\frac{3}{4} S^{--} (\bar{\mathcal{D}}^-)^2 + \frac{3}{4} \bar{S}^{--} (\mathcal{D}^-)^2 \right] \mathcal{L}^{++} \Big| . \quad (4.33)$$

Its variation is

$$\begin{aligned} \delta S_1 = \oint d\mu^{(-2,-4)} \beta \int d^4x e \left[\frac{3}{2} S^{+-} (\bar{\mathcal{D}}^-)^2 + \frac{3}{2} \bar{S}^{+-} (\mathcal{D}^-)^2 \right. \\ \left. - 12S^{--} \bar{S}^{+-} - 12\bar{S}^{--} S^{+-} \right] \mathcal{L}^{++} \Big| , \end{aligned} \quad (4.34)$$

and therefore the functional $S_0 + S_1$ varies as

$$\begin{aligned} \delta(S_0 + S_1) = \oint d\mu^{(-2,-4)} \beta \int d^4x e \left[-\frac{i}{2} (u^+u^-) \mathcal{D}_{\alpha\dot{\alpha}} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} - 3(\bar{\mathcal{D}}_\alpha^- S^{+-}) \bar{\mathcal{D}}^{\dot{\alpha}-} \right. \\ \left. - 3(\mathcal{D}^{\alpha-} \bar{S}^{+-}) \mathcal{D}_\alpha^- - (u^+u^-) (\bar{\mathcal{D}}_\alpha^- \bar{W}^{\dot{\alpha}\dot{\delta}}) \bar{\mathcal{D}}_{\dot{\delta}}^- + (u^+u^-) (\mathcal{D}^{\alpha-} W_{\alpha\beta}) \mathcal{D}^{\beta-} \right. \\ \left. - 18S^{--} \bar{S}^{+-} - 18S^{+-} \bar{S}^{--} \right] \mathcal{L}^{++} \Big| . \end{aligned} \quad (4.35)$$

To cancel the variation in the last line, we have to add to the action another term

$$S_2 = \oint d\mu^{(-2,-4)} \int d^4x e \left[9S^{--} \bar{S}^{--} \right] \mathcal{L}^{++} \Big| . \quad (4.36)$$

As a result, the functional $S_0 + S_1 + S_2$ varies as

$$\begin{aligned} \delta(S_0 + S_1 + S_2) = \oint d\mu^{(-2,-4)} \beta \int d^4x e \left[-\frac{i}{2} (u^+u^-) \mathcal{D}_{\alpha\dot{\alpha}} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} - 3(\bar{\mathcal{D}}_\alpha^- S^{+-}) \bar{\mathcal{D}}^{\dot{\alpha}-} \right. \\ \left. - 3(\mathcal{D}^{\alpha-} \bar{S}^{+-}) \mathcal{D}_\alpha^- - (u^+u^-) (\bar{\mathcal{D}}_\alpha^- \bar{W}^{\dot{\alpha}\dot{\delta}}) \bar{\mathcal{D}}_{\dot{\delta}}^- + (u^+u^-) (\mathcal{D}^{\alpha-} W_{\alpha\beta}) \mathcal{D}^{\beta-} \right] \mathcal{L}^{++} \Big| . \end{aligned} \quad (4.37)$$

In the first term of the variation obtained, we can make use of (4.4). This leads to

$$\oint d\mu^{(-2,-4)} \beta \int d^4x e \left[-\frac{i}{2} (u^+u^-) \mathcal{D}_{\alpha\dot{\alpha}} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} \right] \mathcal{L}^{++} \Big|$$

$$\begin{aligned}
 &= \oint d\mu^{(-2,-4)\beta} \int d^4x e \left[-\frac{i}{2}(u^+u^-) \left(\nabla_{\alpha\dot{\alpha}} - \frac{1}{(u^+u^-)} \Psi_{\alpha\dot{\alpha}}^{\gamma+} \mathcal{D}_{\dot{\gamma}}^- + \frac{1}{(u^+u^-)} \Psi_{\alpha\dot{\alpha}}^{\gamma-} \mathcal{D}_{\dot{\gamma}}^+ \right. \right. \\
 &\quad \left. \left. - \frac{1}{(u^+u^-)} \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^+ \bar{\mathcal{D}}^{\dot{\gamma}-} + \frac{1}{(u^+u^-)} \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^- \bar{\mathcal{D}}^{\dot{\gamma}+} + \phi_{\alpha\dot{\alpha}}^{kl} J_{kl} \right) \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} \right] \mathcal{L}^{++}. \quad (4.38)
 \end{aligned}$$

This variation can be simplified, in complete analogy with the above calculation, by systematically moving the Lorentz and SU(2) generators as well as the derivatives \mathcal{D}^+ , $\bar{\mathcal{D}}^+$ to the right until they hit \mathcal{L}^{++} , at which stage we can use the identity $M_{ab}\mathcal{L}^{++} = 0$, eq. (4.27) and the analyticity conditions $\mathcal{D}_{\alpha}^+\mathcal{L}^{++} = \bar{\mathcal{D}}_{\dot{\alpha}}^+\mathcal{L}^{++} = 0$. We then find

$$\begin{aligned}
 &\oint d\mu^{(-2,-4)\beta} \int d^4x e \left[-\frac{i}{2}(u^+u^-) \mathcal{D}_{\alpha\dot{\alpha}} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} \right] \mathcal{L}^{++} \\
 &= \oint d\mu^{(-2,-4)\beta} \int d^4x e \left[-\frac{i}{2}(u^+u^-) \nabla_{\alpha\dot{\alpha}} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} - \frac{i}{4} \Psi_{\alpha\dot{\alpha}}^{\alpha+} (\mathcal{D}^-)^2 \bar{\mathcal{D}}_{\dot{\alpha}}^- - \frac{i}{4} \bar{\Psi}_{\alpha\dot{\alpha}}^{\alpha+} (\bar{\mathcal{D}}^-)^2 \mathcal{D}_{\dot{\alpha}}^- \right. \\
 &\quad + 2\phi_{\alpha\dot{\alpha}}^{+-} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} + \phi_{\alpha\dot{\alpha}}^{--} \{ \mathcal{D}^{\alpha+}, \bar{\mathcal{D}}^{\dot{\alpha}-} \} + (u^+u^-) \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} \mathcal{D}_{\dot{\gamma}\dot{\alpha}} \mathcal{D}_{\alpha}^- + (u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} \mathcal{D}_{\alpha\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\alpha}}^- \\
 &\quad + 3i(u^+u^-) \Psi_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} Y_{\alpha\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\alpha}}^- - 4i \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} S^{+-} \bar{\mathcal{D}}_{\dot{\alpha}}^- + 4i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} G_{\alpha\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\alpha}}^- \\
 &\quad - 2i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} G_{\alpha\dot{\gamma}} \bar{\mathcal{D}}^{\dot{\gamma}-} + 3i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} \bar{Y}_{\dot{\alpha}\dot{\gamma}} \mathcal{D}_{\alpha}^- - 4i \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} \bar{S}^{+-} \mathcal{D}_{\alpha}^- \\
 &\quad - 4i(u^+u^-) \Psi_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} G_{\dot{\gamma}\dot{\alpha}} \mathcal{D}_{\alpha}^- + 2i(u^+u^-) \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} G_{\beta\dot{\alpha}} \mathcal{D}^{\beta-} + 3i(u^+u^-) \Psi_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} (\bar{\mathcal{D}}_{\dot{\alpha}}^- Y_{\alpha\dot{\gamma}}) \\
 &\quad + 3i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} (\mathcal{D}_{\alpha}^- \bar{Y}_{\dot{\alpha}\dot{\gamma}}) - i(u^+u^-) \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} (\bar{\mathcal{D}}^{\dot{\delta}-} \bar{W}_{\dot{\alpha}\dot{\delta}}) - i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} (\mathcal{D}^{-\delta} W_{\gamma\delta}) \\
 &\quad \left. - 3i \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} (\bar{\mathcal{D}}_{\dot{\alpha}}^- S^{+-}) - 3i \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} (\mathcal{D}_{\alpha}^- \bar{S}^{+-}) \right] \mathcal{L}^{++}. \quad (4.39)
 \end{aligned}$$

Now, in order to cancel out the second, third, fourth and fifth terms, we have to add to the action one more term

$$\begin{aligned}
 S_3 = \oint d\mu^{(-2,-4)} \int d^4x e \left[\frac{i}{4} \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} (\mathcal{D}^-)^2 \bar{\mathcal{D}}_{\dot{\alpha}}^- + \frac{i}{4} \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} (\bar{\mathcal{D}}^-)^2 \mathcal{D}_{\dot{\alpha}}^- \right. \\
 \left. - \phi_{\alpha\dot{\alpha}}^{--} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} \right] \mathcal{L}^{++}. \quad (4.40)
 \end{aligned}$$

Evaluating the variation of S_3 and combining it with $\delta(S_0 + S_1 + S_2)$ gives

$$\begin{aligned}
 \delta(S_0 + S_1 + S_2 + S_3) = \oint d\mu^{(-2,-4)\beta} \int d^4x e \left[-\frac{i}{2}(u^+u^-) \nabla_{\alpha\dot{\alpha}} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} \right. \\
 + (u^+u^-) \Psi_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} \mathcal{D}_{\dot{\gamma}\dot{\alpha}} \mathcal{D}_{\alpha}^- + (u^+u^-) \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} \mathcal{D}_{\beta\dot{\alpha}} \mathcal{D}^{\beta-} + (u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} \mathcal{D}_{\alpha\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\alpha}}^- \\
 + (u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} \mathcal{D}_{\alpha\dot{\beta}} \bar{\mathcal{D}}^{\dot{\beta}-} + 3i(u^+u^-) \Psi_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} Y_{\alpha\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\alpha}}^- - 9i \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} S^{+-} \bar{\mathcal{D}}_{\dot{\alpha}}^- \\
 + 4i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} G_{\alpha\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\alpha}}^- - i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} G_{\alpha\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\gamma}-} - i(u^+u^-) \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} \bar{W}_{\dot{\gamma}\dot{\alpha}} \bar{\mathcal{D}}^{\dot{\gamma}-} \\
 - 3(\bar{\mathcal{D}}_{\dot{\alpha}}^- S^{+-}) \bar{\mathcal{D}}^{\dot{\alpha}-} - (u^+u^-) (\bar{\mathcal{D}}_{\dot{\alpha}}^- \bar{W}^{\dot{\alpha}\dot{\delta}}) \bar{\mathcal{D}}_{\dot{\delta}}^- + 3i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} \bar{Y}_{\dot{\alpha}\dot{\gamma}} \mathcal{D}_{\alpha}^- - 9i \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} \bar{S}^{+-} \mathcal{D}_{\alpha}^- \\
 - 4i(u^+u^-) \Psi_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} G_{\dot{\gamma}\dot{\alpha}} \mathcal{D}_{\alpha}^- + i(u^+u^-) \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} G_{\alpha\dot{\alpha}} \mathcal{D}^{\beta-} - i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} W_{\alpha\dot{\gamma}} \mathcal{D}^{\gamma-} \\
 - 3(\mathcal{D}^{\alpha-} \bar{S}^{+-}) \mathcal{D}_{\alpha}^- + (u^+u^-) (\mathcal{D}^{\alpha-} W_{\alpha\beta}) \mathcal{D}^{\beta-} + 3i(u^+u^-) \Psi_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} (\bar{\mathcal{D}}_{\dot{\alpha}}^- Y_{\alpha\dot{\gamma}}) \\
 + 3i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} (\mathcal{D}_{\alpha}^- \bar{Y}_{\dot{\alpha}\dot{\gamma}}) - 3i(u^+u^-) \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} (\bar{\mathcal{D}}^{\dot{\beta}-} \bar{W}_{\dot{\alpha}\dot{\beta}}) - 3i(u^+u^-) \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} (\mathcal{D}^{-\beta} W_{\alpha\beta}) \\
 \left. - 9i \Psi_{\alpha\dot{\alpha}}^{\alpha\dot{\gamma}-} (\bar{\mathcal{D}}_{\dot{\alpha}}^- S^{+-}) - 9i \bar{\Psi}_{\alpha\dot{\alpha}\dot{\gamma}}^{\alpha\dot{\gamma}-} (\mathcal{D}_{\alpha}^- \bar{S}^{+-}) \right] \mathcal{L}^{++}. \quad (4.41)
 \end{aligned}$$

Let us consider the first to fifth terms in (4.41) which involve vector covariant derivatives. In this sector, we apply (4.4), the formula for integration by parts, eq. (4.10), with the space-time torsion (4.12a) expressed as

$$\mathcal{T}_{ab}{}^c = -\frac{4i}{(u^+u^-)} \left(\Psi_{[a}{}^{\gamma+} \bar{\Psi}_{b]\delta}{}^- (\sigma^c)_{\gamma}{}^{\delta} - \Psi_{[a}{}^{\gamma-} \bar{\Psi}_{b]\delta}{}^+ (\sigma^c)_{\gamma}{}^{\delta} \right). \quad (4.42)$$

Implementing also the usual iterative procedure, we obtain

$$\begin{aligned} & \oint d\mu^{(-2,-4)} \beta \int d^4x e (u^+u^-) \left[-\frac{i}{2} \nabla_{\alpha\dot{\alpha}} \mathcal{D}^{\alpha-} \bar{\mathcal{D}}^{\dot{\alpha}-} + \Psi^{\alpha\dot{\alpha}\beta-} \mathcal{D}_{\beta\dot{\alpha}} \mathcal{D}_{\alpha}{}^- + \Psi^{\alpha\dot{\alpha}-} \mathcal{D}_{\beta\dot{\alpha}} \mathcal{D}^{\beta-} \right. \\ & \quad \left. + \bar{\Psi}^{\alpha\dot{\alpha}\dot{\gamma}-} \mathcal{D}_{\alpha\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\alpha}}{}^- + \bar{\Psi}^{\alpha\dot{\alpha}-} \mathcal{D}_{\alpha\dot{\beta}} \bar{\mathcal{D}}^{\dot{\beta}-} \right] \mathcal{L}^{++} \Big| \\ &= \oint d\mu^{(-2,-4)} \beta \int d^4x e \left[2(\sigma^{ab})^{\alpha}{}_{\beta} \Psi_{[a}{}^{\beta+} \bar{\Psi}_{b]\dot{\alpha}}{}^- \mathcal{D}_{\alpha}{}^- \bar{\mathcal{D}}_{\dot{\alpha}}{}^- + 2(\tilde{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \Psi_{[a}{}^{\alpha+} \bar{\Psi}_{b]\dot{\beta}}{}^- \mathcal{D}_{\alpha}{}^- \bar{\mathcal{D}}_{\dot{\alpha}}{}^- \right. \\ & \quad + 2(\sigma^{ab})^{\alpha}{}_{\beta} \Psi_{[a}{}^{\beta-} \bar{\Psi}_{b]\dot{\alpha}}{}^+ \mathcal{D}_{\alpha}{}^- \bar{\mathcal{D}}_{\dot{\alpha}}{}^- + 2(\tilde{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \Psi_{[a}{}^{\alpha-} \bar{\Psi}_{b]\dot{\beta}}{}^+ \mathcal{D}_{\alpha}{}^- \bar{\mathcal{D}}_{\dot{\alpha}}{}^- - 2(\sigma^{ab})^{\alpha\beta} \Psi_{a\alpha}{}^- \Psi_{b\beta}{}^+ (\mathcal{D}^-)^2 \\ & \quad - 2(\tilde{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{\Psi}_{a\dot{\alpha}}{}^- \bar{\Psi}_{b\dot{\beta}}{}^+ (\bar{\mathcal{D}}^-)^2 - 4(u^+u^-) (\sigma^{ab})_{\beta}{}^{\gamma} \mathcal{T}_{[a|c]}{}^c \Psi_{b]}{}^{\beta-} \mathcal{D}_{\gamma}{}^- \\ & \quad + 4(u^+u^-) (\tilde{\sigma}^{ab})_{\dot{\beta}}{}^{\dot{\gamma}} \mathcal{T}_{[a|c]}{}^c \bar{\Psi}_{b]}{}^{\dot{\beta}-} \bar{\mathcal{D}}_{\dot{\gamma}}{}^- + 4(u^+u^-) (\sigma^{ab})_{\beta}{}^{\gamma} (\nabla_{[a} \Psi_{b]}{}^{\beta-}) \mathcal{D}_{\gamma}{}^- \\ & \quad - 4(u^+u^-) (\tilde{\sigma}^{ab})_{\dot{\beta}}{}^{\dot{\gamma}} (\nabla_{[a} \bar{\Psi}_{b]}{}^{\dot{\beta}-}) \bar{\mathcal{D}}_{\dot{\gamma}}{}^- + 12(\sigma^{ab})_{\beta}{}^{\gamma} \Psi_a{}^{\beta-} \phi_b{}^{+-} \mathcal{D}_{\gamma}{}^- - 12(\tilde{\sigma}^{ab})_{\dot{\beta}}{}^{\dot{\gamma}} \bar{\Psi}_a{}^{\dot{\beta}-} \phi_b{}^{+-} \bar{\mathcal{D}}_{\dot{\gamma}}{}^- \\ & \quad + 4(\sigma^{ab})^{\alpha}{}_{\beta} \Psi_a{}^{\beta-} \bar{\Psi}_b{}^{\dot{\gamma}-} \{ \bar{\mathcal{D}}_{\dot{\gamma}}{}^+, \mathcal{D}_{\alpha}{}^- \} - 4(\tilde{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\Psi}_a{}^{\dot{\beta}-} \Psi_b{}^{\gamma-} \{ \mathcal{D}_{\gamma}{}^+, \bar{\mathcal{D}}_{\dot{\alpha}}{}^- \} \\ & \quad + 4(\sigma^{ab})^{\alpha}{}_{\beta} \Psi_a{}^{\beta-} \Psi_b{}^{\gamma-} \{ \mathcal{D}_{\gamma}{}^+, \mathcal{D}_{\alpha}{}^- \} - 4(\tilde{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\Psi}_a{}^{\dot{\beta}-} \bar{\Psi}_b{}^{\dot{\gamma}-} \{ \bar{\mathcal{D}}_{\dot{\gamma}}{}^+, \bar{\mathcal{D}}_{\dot{\alpha}}{}^- \} \\ & \quad - 32(u^+u^-) (\sigma^{ab})^{\alpha}{}_{\beta} \Psi_a{}^{\beta-} \bar{\Psi}_b{}^{\dot{\alpha}-} G_{\alpha\dot{\alpha}} - 8(u^+u^-) (\sigma^{ab})^{\alpha}{}_{\beta} \Psi_a{}^{\beta-} \Psi_b{}^{\gamma-} Y_{\alpha\gamma} \\ & \quad \left. - 8(u^+u^-) (\tilde{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{\Psi}_a{}^{\dot{\beta}-} \bar{\Psi}_b{}^{\dot{\gamma}-} \bar{Y}_{\dot{\alpha}\dot{\gamma}} \right] \mathcal{L}^{++} \Big|. \quad (4.43) \end{aligned}$$

In order to cancel the first six terms in (4.43), we have to add to the action one more structure

$$\begin{aligned} S_4 = & \oint d\mu^{(-2,-4)} \int d^4x e \left[-2(\sigma^{ab})^{\alpha}{}_{\beta} \Psi_{[a}{}^{\beta-} \bar{\Psi}_{b]\dot{\alpha}}{}^- \mathcal{D}_{\alpha}{}^- \bar{\mathcal{D}}_{\dot{\alpha}}{}^- - 2(\tilde{\sigma}^{ab})^{\dot{\alpha}}{}_{\dot{\beta}} \Psi_{[a}{}^{\alpha-} \bar{\Psi}_{b]\dot{\beta}}{}^- \mathcal{D}_{\alpha}{}^- \bar{\mathcal{D}}_{\dot{\alpha}}{}^- \right. \\ & \left. + (\sigma^{ab})^{\alpha\beta} \Psi_{a\alpha}{}^- \Psi_{b\beta}{}^- (\mathcal{D}^-)^2 + (\tilde{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}} \bar{\Psi}_{a\dot{\alpha}}{}^- \bar{\Psi}_{b\dot{\beta}}{}^- (\bar{\mathcal{D}}^-)^2 \right] \mathcal{L}^{++}(z, u^+) \Big| \quad (4.44) \end{aligned}$$

and consider the variation $\delta(S_0 + S_1 + S_2 + S_3 + S_4)$. We use (4.4), then move \mathcal{D}^+ , $\bar{\mathcal{D}}^+$ derivatives, Lorentz and SU(2) generators to the right. Next we should move to the left all ∇_a derivatives and use the rule for integration by parts, eq. (4.10). At this stage, we can use the identities

$$\begin{aligned} \nabla_{[a} \Psi_{b]}{}^{\gamma-} &= -\frac{1}{8} (\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} (\mathcal{D}^{\gamma-} \bar{Y}_{\dot{\alpha}\dot{\beta}}) + \frac{1}{8} (\sigma_{ab})^{\alpha\beta} (\mathcal{D}^{\gamma-} W_{\alpha\beta}) + \frac{1}{4} (\sigma_{ab})^{\gamma\delta} \mathcal{D}_{\delta}{}^- \bar{S}^{+-} \\ & \quad + i(\sigma_{[a})^{\alpha}{}_{\beta} \Psi_{b]\dot{\alpha}}{}^- G^{\gamma\dot{\beta}} \Big| - \frac{i}{2(u^+u^-)} (\tilde{\sigma}_{[a})^{\dot{\alpha}\dot{\gamma}} \bar{\Psi}_{b]\dot{\alpha}}{}^+ \bar{S}^{--} \Big| + \frac{i}{2(u^+u^-)} (\tilde{\sigma}_{[a})^{\dot{\alpha}\dot{\gamma}} \bar{\Psi}_{b]\dot{\alpha}}{}^- \bar{S}^{+-} \Big| \\ & \quad - \frac{i}{2} (\sigma_{[a})_{\alpha}{}^{\dot{\alpha}} \bar{\Psi}_{b]\dot{\alpha}}{}^- W^{\alpha\gamma} \Big| - \frac{i}{2} (\sigma_{[a})^{\gamma}{}_{\beta} \bar{\Psi}_{b]\dot{\alpha}}{}^- \bar{Y}^{\dot{\alpha}\dot{\beta}} \Big| + \frac{1}{(u^+u^-)} \left\{ \phi_{[a}{}^{+-} \Psi_{b]}{}^{\gamma-} - \phi_{[a}{}^{--} \Psi_{b]}{}^{\gamma+} \right\} \\ & \quad - \frac{2i}{(u^+u^-)} (\sigma^c)_{\delta}{}^{\dot{\delta}} \Psi_c{}^{\gamma-} \left\{ \Psi_{[a}{}^{\delta+} \bar{\Psi}_{b]\dot{\delta}}{}^- - \Psi_{[a}{}^{\delta-} \bar{\Psi}_{b]\dot{\delta}}{}^+ \right\} \quad (4.45) \end{aligned}$$

and

$$\begin{aligned}
 \nabla_{[a}\bar{\Psi}_{b]\dot{\gamma}}^- &= -\frac{1}{8}(\sigma_{ab})^{\alpha\beta}(\bar{\mathcal{D}}_{\dot{\gamma}}^- Y_{\alpha\beta})| + \frac{1}{8}(\tilde{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}}(\bar{\mathcal{D}}_{\dot{\gamma}}^- \bar{W}_{\dot{\alpha}\dot{\beta}})| - \frac{1}{4}(\tilde{\sigma}^{ab})_{\dot{\gamma}\dot{\delta}}(\bar{\mathcal{D}}^{\dot{\delta}-} S^{+-})| \\
 &- i(\sigma_{[a}{}^\alpha{}_{(\dot{\alpha}}\bar{\Psi}_{b]}^{\dot{\alpha}-} G_{\alpha\dot{\gamma}})| + \frac{i}{2(u^+u^-)}(\sigma_{[a}{}_{\alpha\dot{\gamma}}\Psi_{b]}^{\alpha+} S^{--})| - \frac{i}{2(u^+u^-)}(\sigma_{[a}{}_{\alpha\dot{\gamma}}\Psi_{b]}^{\alpha-} S^{+-})| \\
 &- \frac{i}{2}(\sigma_{[a}{}_{\alpha}{}^{\dot{\delta}}\bar{\Psi}_{b]}^{\alpha-} \bar{W}_{\dot{\delta}\dot{\gamma}}| - \frac{i}{2}(\sigma_{[a}{}^\beta{}_{\dot{\gamma}}\Psi_{b]}^{\alpha-} Y_{\alpha\beta})| + \frac{1}{(u^+u^-)}\left\{\phi_{[a}{}^{+-}\bar{\Psi}_{b]\dot{\gamma}}^- - \phi_{[a}{}^{--}\bar{\Psi}_{b]\dot{\gamma}}^+\right\} \\
 &- \frac{2i}{(u^+u^-)}(\sigma^c)_{\delta}{}^{\dot{\delta}}\left\{\Psi_{[a}{}^{\delta+}\bar{\Psi}_{b]\dot{\delta}}^- - \Psi_{[a}{}^{\delta-}\bar{\Psi}_{b]\dot{\delta}}^+\right\}\bar{\Psi}_{c\dot{\gamma}}^-, \tag{4.46}
 \end{aligned}$$

which follow from (4.12b) and (4.12c). After rather long computation, which involves algebraic manipulations using some results from appendix A, non-trivial cancellations occur. One obtains

$$\begin{aligned}
 \delta(S_0 + S_1 + S_2 + S_3 + S_4) &= \oint d\mu^{(-2,-4)}\beta \int d^4x e \left[-24(\sigma^{ab})^{\alpha\beta}\Psi_{a\alpha}^-\Psi_{b\beta}^- S^{+-} \right. \\
 &- 24(\sigma^{ab})_{\alpha\beta}\Psi_a^{\alpha+}\Psi_b^{\beta-} S^{--} - 24(\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}\bar{\Psi}_a^{\dot{\alpha}-}\bar{\Psi}_b^{\dot{\beta}-} \bar{S}^{+-} - 24(\tilde{\sigma}^{cd})_{\dot{\alpha}\dot{\beta}}\bar{\Psi}_a^{\dot{\beta}+}\bar{\Psi}_b^{\dot{\alpha}-} \bar{S}^{--} \\
 &- 6i\bar{\Psi}^{\alpha\dot{\alpha}-}\bar{S}^{+-}\mathcal{D}_{\dot{\alpha}}^- - 3i\bar{\Psi}^{\alpha\dot{\alpha}-}\bar{S}^{--}\mathcal{D}^{\alpha-} - 6i\Psi^{\alpha\dot{\alpha}-}S^{+-}\bar{\mathcal{D}}_{\dot{\alpha}}^- - 3i\Psi^{\alpha\dot{\alpha}-}S^{--}\bar{\mathcal{D}}_{\dot{\alpha}}^- \\
 &+ 8(\sigma^{ab})_{\alpha\beta}\phi_a^{+-}\Psi_b^{\alpha-}\mathcal{D}^{\beta-} + 4(\sigma^{ab})_{\alpha\beta}\phi_a^{--}\Psi_b^{\alpha+}\mathcal{D}^{\beta-} - 8(\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}\phi_a^{+-}\bar{\Psi}_b^{\dot{\alpha}-}\bar{\mathcal{D}}^{\dot{\beta}-} \\
 &- 4(\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}}\phi_a^{--}\bar{\Psi}_b^{\dot{\alpha}+}\bar{\mathcal{D}}^{\dot{\beta}-} - 4\varepsilon^{abcm}(\sigma_m)_{\alpha\dot{\alpha}}\Psi_a^{\alpha+}\Psi_b^{\beta-}\bar{\Psi}_c^{\dot{\alpha}-}\mathcal{D}_{\dot{\beta}}^- \\
 &- 4\varepsilon^{abcm}(\sigma_m)_{\alpha\dot{\alpha}}\Psi_a^{\beta-}\Psi_b^{\alpha-}\bar{\Psi}_c^{\dot{\alpha}+}\mathcal{D}_{\dot{\beta}}^- - 4\varepsilon^{abcm}(\sigma_m)_{\alpha\dot{\alpha}}\Psi_a^{\alpha-}\Psi_b^{\beta+}\bar{\Psi}_c^{\dot{\alpha}-}\mathcal{D}_{\dot{\beta}}^- \\
 &- 4\varepsilon^{abcm}(\sigma_m)_{\alpha\dot{\alpha}}\Psi_a^{\alpha+}\bar{\Psi}_b^{\dot{\alpha}-}\bar{\Psi}_c^{\dot{\beta}-}\bar{\mathcal{D}}_{\dot{\beta}}^- - 4\varepsilon^{abcm}(\sigma_m)_{\alpha\dot{\alpha}}\Psi_a^{\alpha-}\bar{\Psi}_b^{\dot{\alpha}+}\bar{\Psi}_c^{\dot{\beta}-}\bar{\mathcal{D}}_{\dot{\beta}}^- \\
 &- 4\varepsilon^{abcm}(\sigma_m)_{\alpha\dot{\alpha}}\Psi_a^{\alpha-}\bar{\Psi}_b^{\dot{\alpha}-}\bar{\Psi}_c^{\dot{\beta}+}\bar{\mathcal{D}}_{\dot{\beta}}^- + 12\varepsilon^{abcm}(\sigma_m)_{\alpha\dot{\alpha}}\phi_a^{--}\Psi_b^{\alpha+}\bar{\Psi}_c^{\dot{\alpha}-} \\
 &+ 12\varepsilon^{abcm}(\sigma_m)_{\alpha\dot{\alpha}}\phi_a^{--}\Psi_b^{\alpha-}\bar{\Psi}_c^{\dot{\alpha}+} + 24\varepsilon^{abcm}(\sigma_m)_{\alpha\dot{\alpha}}\phi_a^{+-}\Psi_b^{\alpha-}\bar{\Psi}_c^{\dot{\alpha}-}\left.\right] \mathcal{L}^{++} \Big|. \tag{4.47}
 \end{aligned}$$

The nontrivial point is that all terms with four gravitinos have identically cancelled out at this stage. And one more iteration – we have to add to the action the following structure:

$$\begin{aligned}
 S_5 &= \oint d\mu^{(-2,-4)} \int d^4x e \left[3i\bar{\Psi}^{\alpha\dot{\alpha}-}\bar{S}^{--}\mathcal{D}_{\dot{\alpha}}^- + 3i\Psi^{\alpha\dot{\alpha}-}S^{--}\bar{\mathcal{D}}_{\dot{\alpha}}^- + 12(\sigma^{ab})^{\alpha\beta}\Psi_{a\alpha}^-\Psi_{b\beta}^- S^{--} \right. \\
 &+ 12(\tilde{\sigma}^{ab})^{\dot{\alpha}\dot{\beta}}\bar{\Psi}_{a\dot{\alpha}}^-\bar{\Psi}_{b\dot{\beta}}^-\bar{S}^{--} - 4(\sigma^{ab})_{\beta\gamma}\phi_a^{--}\Psi_b^{\gamma-}\mathcal{D}^{\beta-} + 4(\tilde{\sigma}^{ab})^{\dot{\beta}\dot{\gamma}}\phi_a^{--}\bar{\Psi}_{b\dot{\gamma}}^-\bar{\mathcal{D}}_{\dot{\beta}}^- \\
 &- 12\varepsilon^{abcd}(\sigma_d)_{\gamma\dot{\alpha}}\phi_a^{--}\Psi_b^{\gamma-}\bar{\Psi}_c^{\dot{\alpha}-} + 4\varepsilon^{abcd}(\sigma_d)_{\alpha\dot{\alpha}}\Psi_a^{\alpha-}\Psi_b^{\beta-}\bar{\Psi}_c^{\dot{\alpha}-}\mathcal{D}_{\dot{\beta}}^- \\
 &+ 4\varepsilon^{abcd}(\sigma_d)_{\alpha\dot{\alpha}}\Psi_a^{\alpha-}\bar{\Psi}_b^{\dot{\alpha}-}\bar{\Psi}_c^{\dot{\beta}-}\bar{\mathcal{D}}_{\dot{\beta}}^- \left.\right] \mathcal{L}^{++} \Big|. \tag{4.48}
 \end{aligned}$$

This proves to complete the procedure. One can now check that

$$\delta(S_0 + S_1 + S_2 + S_3 + S_4 + S_5) = \delta S = 0 \tag{4.49}$$

We have thus demonstrated that the action (4.13) is uniquely obtained from the requirement of projective invariance.

4.3 Analysis of the results

The component action (4.13) is the main result of this work. In technical terms, our procedure for deriving (4.13) from the original superspace action (1.2) has many similarities with the earlier construction for 5D $\mathcal{N} = 1$ supergravity [2]. There is, however, a very important conceptual difference. The point is that, unlike the five dimensional analysis in [2], no Wess-Zumino gauge has been assumed in the process of deriving (4.13).¹⁵ In other words, all the gauge symmetries of the parental superspace action (1.2) are preserved by its component counterpart (4.13).¹⁶ This huge gauge freedom can be used at will depending on concrete dynamical circumstances. It is worth giving two examples.

The action is invariant under the super-Weyl transformations generated by a covariantly chiral parameter σ , $\bar{\mathcal{D}}_{\dot{\alpha}}^i \sigma = 0$. This local symmetry can be used to choose a useful gauge condition, for instance, to set the field strength W of the compensating vector multiplet to be

$$W = 1 . \tag{4.50}$$

The action is invariant under local SU(2) transformations generated by a real symmetric parameter K^{ij} that is otherwise unconstrained, see eqs. (A.5) and (A.6). Consider an off-shell tensor multiplet described by a symmetric real superfield $H^{ij}(z)$,

$$\mathcal{D}_{\alpha}^{(i} H^{jk)} = \bar{\mathcal{D}}_{\dot{\alpha}}^{(i} H^{jk)} = 0, \quad H^{ij} = H^{ji}, \quad \overline{H^{ij}} = H_{ij} . \tag{4.51}$$

Associated with $H^{ij}(z)$ is the $O(2)$ multiplet $H^{++}(z, u^+) = H^{ij}(z) u_i^+ u_j^+$. We will assume H^{ij} to be nowhere vanishing,

$$H^{ij} H_{ij} \neq 0, \tag{4.52}$$

the condition required of a superconformal compensator. Then, the SU(2) gauge freedom can be partially fixed as

$$H^{ij} = -\frac{i}{2} (\sigma_1)^{ij} G, \quad \bar{G} = G > 0, \quad (\sigma_1)^{ij} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{4.53}$$

which leaves an unbroken U(1) gauge symmetry. To be consistent with the constraint (4.51), the SU(2) connection should be

$$\Phi_{\alpha}^{i j k} = i \Sigma_{\alpha}^i (\sigma_1)^{jk} + \varepsilon^{i(j} E_{\alpha}^{k)} \ln G, \tag{4.54}$$

with Σ_{α}^i a U(1) connection. We will give an application of the gauge condition (4.53) in the next subsection.

Let us denote by $\mathcal{P}^{(0,4)}(u^-)$ the differential operator in the square brackets in (4.13). Then the component action can be rewritten as

$$S = \int d^4x e \oint_C d\mu^{(-2,-4)} \mathcal{P}^{(0,4)}(u^-) \mathcal{L}^{++}(z, u^+) | . \tag{4.55}$$

¹⁵A careful analysis of the 5D construction [2] shows that the choice of the Wess-Zumino gauge was not essential. It is just an unfortunate stereotype forced upon us by textbook lessons [10–12] that choosing Wess-Zumino gauge is imperative for component reduction.

¹⁶Most of purely gauge degrees of freedom are contained in the vielbein and connection superfields for \mathcal{D}_{α}^i and $\bar{\mathcal{D}}_{\dot{\alpha}}^i$. In the construction used, however, these objects do not show up explicitly.

Without loss of generality, we can assume the north pole of $\mathbb{C}P^1$, i.e. $u^{+i} \propto (0, 1)$, to be outside of the integration contour, hence u^+ can be represented as

$$u^{+i} = u^{+1}(1, \zeta) = u^{+1}\zeta^i, \quad \zeta^i = (1, \zeta), \quad \zeta_i = \varepsilon_{ij}\zeta^j = (-\zeta, 1), \quad (4.56)$$

with ζ the local complex coordinate for $\mathbb{C}P^1$. Now, the projective invariance, eqs. (4.18) and (4.19), can be used to set

$$u_i^- \equiv \hat{u}_i^- = (1, 0), \quad \hat{u}^{-i} = \varepsilon^{ij}\hat{u}_j^- = (0, -1). \quad (4.57)$$

Finally, representing the Lagrangian in the form

$$\mathcal{L}^{++}(z, u^+) = i u^{+1} u^{+2} \mathcal{L}(z, \zeta) = i (u^{+1})^2 \zeta \mathcal{L}(z, \zeta), \quad (4.58)$$

the action turns into

$$S = - \int d^4x e \mathcal{P} \oint_C \frac{d\zeta}{2\pi i} \zeta \mathcal{L}(z, \zeta), \quad \mathcal{P} := \mathcal{P}^{(0,4)}(\hat{u}^-). \quad (4.59)$$

The important point is that the operator \mathcal{P} is ζ -independent, and therefore its presence is not relevant when evaluating the contour integral. If the original Lagrangian, \mathcal{L}^{++} , depends on matter superfields only, the contour integral in (4.59) corresponds to that arising in a rigid superconformal theory [4].

4.4 Application I: gauge invariance of the vector-tensor coupling

Let $S(\mathcal{L}^{++})$ denote the action (1.2). Consider $\mathcal{L}_{v-t}^{++} = H^{++}V$, where $H^{++}(z, u^+)$ is a tensor multiplet (or a real $O(2)$ multiplet), and $V(z, u^+)$ a real weight-zero tropical multiplet (see [1] for more detail). The latter describes a massless vector multiplet provided there is gauge invariance

$$\delta V = \lambda + \tilde{\lambda}, \quad (4.60)$$

where $\lambda(z, u^+)$ is an arctic weight-zero multiplet, and $\tilde{\lambda}(z, u^+)$ its smile conjugate (see [1] for more detail). We can now prove that the functional $S(H^{++}V)$ is invariant under the gauge transformation (4.60). It is sufficient to prove that

$$S(H^{++}\lambda) = 0, \quad (4.61)$$

for an arbitrary arctic weight-zero superfield $\lambda(z, u^+)$. The latter follows from the fact that the action (4.13) with $\mathcal{L}^{++} = H^{++}\lambda$ has no pole in the complex ζ -plane.

4.5 Application II: the c-map

In this subsection we would like to give a curved superspace description for the c-map [41, 42]. The problem of developing a superspace description for the c-map has already been discussed in [47] (see also [48]) and [49]. However, since no projective superspace formulation for 4D $\mathcal{N} = 2$ matter-coupled supergravity was available at that time, the only possible approach to address the problem was

- (i) to use the existence of a one-to-one correspondence between $4n$ -dimensional quaternionic Kähler spaces and $4(n + 1)$ -dimensional hyperkähler manifolds possessing a homothetic Killing vector, and the fact that such hyperkähler spaces (or “hyperkähler cones” [52]) are the target spaces for rigid $\mathcal{N} = 2$ superconformal sigma models; and
- (ii) to construct an appropriate hyperkähler cone associated with a rigid superconformal model of $\mathcal{N} = 2$ tensor multiplets.

Now, we are in a position to overcome all the limitations of the earlier works.

In accordance with [47], a tensor multiplet model corresponding to the c-map is described by the Lagrangian

$$\mathcal{L}^{++} = \frac{1}{2i H_0^{++}} \left(F(H_I^{++}) - \bar{F}(H_I^{++}) \right), \quad I = 1, \dots, N + 1. \quad (4.62)$$

Here H_I^{++} and H_0^{++} are tensor multiplets, with H_0^{++} obeying the constraint (4.52), and $F(z^I)$ is a holomorphic homogeneous function of second degree, $F(cz^I) = c^2 F(z^I)$. Thus we have to consider the following action:

$$S = \text{Im} \int d^4x e \mathcal{P} \oint_C \frac{d\zeta}{2\pi i} \frac{F(H_I(\zeta))}{H_0(\zeta)} \Big|, \quad (4.63)$$

where the superfields $H_I(\zeta)$ and $H_0(\zeta)$ are defined as

$$H_I^{++}(z, u^+) = i(u^{+1})^2 H_I(z, \zeta), \quad H_I(\zeta) = \Phi_I + \zeta G_I - \zeta^2 \bar{\Phi}_I, \quad (4.64)$$

and similarly for $H_0(\zeta)$.

Before we start studying the curved-superspace action (4.63), it is worth giving some comments about its flat superspace version. Let $\mathcal{P}_{\text{flat}}$ and $\mathcal{L}_{\text{flat}}$ be the flat-superspace counterparts of the operator \mathcal{P} (4.59) and the Lagrangian \mathcal{L} (4.58). We obviously have

$$\mathcal{P}_{\text{flat}} = \frac{1}{16} (D^{\underline{1}})^2 (\bar{D}^{\underline{1}})^2 = \frac{1}{16} (D^{\underline{1}})^2 (\bar{D}^{\underline{2}})^2, \quad (4.65)$$

with D_α^i and $\bar{D}_i^{\dot{\alpha}}$ the flat spinor covariant derivatives. It is easy to see that the flat-superspace version of the analyticity conditions (1.1) implies $(\bar{D}_{\underline{1}}^{\dot{\alpha}} + \zeta \bar{D}_{\underline{2}}^{\dot{\alpha}}) \mathcal{L}_{\text{flat}}(\zeta) = 0$, and thus for the rigid supersymmetric action S_{flat} we get

$$\begin{aligned} S_{\text{flat}} &= \text{Im} \int d^4x \oint_C \frac{d\zeta}{2\pi i} \mathcal{P}_{\text{flat}} \frac{F(H_I(\zeta))}{H_0(\zeta)} \Big| = \text{Im} \int d^4x \frac{(D^{\underline{1}})^2 (\bar{D}_{\underline{1}})^2}{16} \oint_C \frac{d\zeta}{2\pi i \zeta^2} \frac{F(H_I(\zeta))}{H_0(\zeta)} \Big| \\ &= \text{Im} \int d^4x d^2\theta d^2\bar{\theta} \oint_C \frac{d\zeta}{2\pi i \zeta^2} \frac{F(H_I(\zeta))}{H_0(\zeta)} \Big|_{\theta_2 = \bar{\theta}^2 = 0} \end{aligned} \quad (4.66)$$

The action obtained defines an $\mathcal{N} = 2$ supersymmetric theory formulated in $\mathcal{N} = 1$ superspace. It is the $\mathcal{N} = 2$ superconformal model which was studied in [47, 49].

In [47], the problem of evaluating the contour integral in (4.66) was reduced to that solved several years earlier in [53] (see also [52]) for the case of the rigid c-map. Specifically, Roček et al. [47] imposed the SU(2) gauge condition (4.53) or, equivalently, $H_0(\zeta) = \zeta G_0$,

which essentially corresponds the rigid c-map (more precisely, $G_0 = 1$ in the case of the rigid c-map, but the presence of G_0 is irrelevant for computing the contour integral). The subtlety with the analysis in [47] is that their tensor multiplet model is rigid superconformal, and hence the $SU(2)$ parameters are constant.¹⁷

In our case, however, the $SU(2)$ transformations are local, and it is legitimate to choose the gauge condition (4.53). As a result, the action turns into

$$S = \text{Im} \int d^4x e^{\mathcal{P}} \frac{F(\Phi_I)}{G_0} \Big| \tag{4.67}$$

provided the contour C in (4.63) is taken to be a circle around the origin in \mathbb{C} . Still, the consideration given is not quite satisfactory, because of a special gauge chosen.

Fortunately, there is no need to impose any $SU(2)$ gauge condition in order to do the contour integral in (4.63). Following the rigid supersymmetric analysis of [49], we represent

$$H_0(\zeta) = -\bar{\Phi}_0 (\zeta - \zeta_+) (\zeta - \zeta_-), \quad \zeta_{\pm} = \frac{1}{2\bar{\Phi}_0} \left(G_0 \mp \sqrt{G_0^2 + 4|\Phi_0|^2} \right) \tag{4.68}$$

and choose the contour C in (4.63) to be a small circle around the root ζ_+ . This leads to

$$S = \text{Im} \int d^4x e^{\mathcal{P}} \frac{F(H_I(\zeta_+))}{\sqrt{G_0^2 + 4|\Phi_0|^2}} \Big|. \tag{4.69}$$

Since

$$\zeta_+ = -\frac{2\Phi_0}{(G_0 + \sqrt{G_0^2 + 4|\Phi_0|^2})} \xrightarrow{\Phi_0 \rightarrow 0} 0, \tag{4.70}$$

the covariant action (4.69) reduces to (4.67) in the limit $\Phi_0 \rightarrow 0$. In the flat superspace limit, we reproduce the results of [47, 49].

5 Chiral representation for the action principle

In this section we derive a new representation for the action principle (1.2) as an integral over the chiral subspace.

The covariantly chiral projector $\bar{\Delta}$ was defined in section 3, eq. (3.23). It turns out that $\bar{\Delta}$ can be given an alternative representation. It is

$$\bar{\Delta} \oint (u^+ du^+) U^{(-2)} = \frac{1}{16} \oint \frac{(u^+ du^+)}{(u^+ u^-)^2} \left((\bar{\mathcal{D}}^-)^2 + 4\bar{S}^{--} \right) \left((\bar{\mathcal{D}}^+)^2 + 4\bar{S}^{++} \right) U^{(-2)}, \tag{5.1}$$

with $U^{(-2)}(z, u^+)$ an arbitrary isotwistor superfield of weight -2 (see [1] for the definition of isotwistor supermultiplets, as well as appendix B below). As before, the constant isotwistor u_i^- is chosen to be linearly independent from u_i^+ , $(u^+ u^-) \neq 0$, but otherwise is completely arbitrary. It is proved in appendix C that that the right-hand side of (5.1)

- (i) remains invariant under arbitrary projective transformations (1.6); and

¹⁷Actually, in the case of rigid $\mathcal{N} = 2$ supersymmetry, if a tensor multiplet is constrained as in eq. (4.53), then it is simply constant, $G = \text{const}$.

(ii) is covariantly chiral.

Let us transform the action functional (1.2) by making use of eqs. (3.22) and (5.1):

$$\begin{aligned}
 S(\mathcal{L}^{++}) &= \frac{1}{2\pi} \int d^4x d^4\theta d^4\bar{\theta} E \oint(u^+ du^+) \frac{W\bar{W}\mathcal{L}^{++}}{(\Sigma^{++})^2} \\
 &= \frac{1}{2\pi} \int d^4x d^4\theta \mathcal{E} \bar{\Delta} \oint(u^+ du^+) \frac{W\bar{W}\mathcal{L}^{++}}{(\Sigma^{++})^2} \\
 &= \frac{1}{32\pi} \int d^4x d^4\theta \mathcal{E} \oint \frac{(u^+ du^+)}{(u^+ u^-)^2} \left((\bar{\mathcal{D}}^-)^2 + 4\bar{S}^{--} \right) \left((\bar{\mathcal{D}}^+)^2 + 4\bar{S}^{++} \right) \frac{W\bar{W}\mathcal{L}^{++}}{(\Sigma^{++})^2} \\
 &= \frac{1}{8\pi} \int d^4x d^4\theta \mathcal{E} W \oint \frac{(u^+ du^+)}{(u^+ u^-)^2} \left((\bar{\mathcal{D}}^-)^2 + 4\bar{S}^{--} \right) \frac{\mathcal{L}^{++}}{\Sigma^{++}}, \tag{5.2}
 \end{aligned}$$

where we have used eq. (1.3), the chirality of the vector multiplet strength, $\bar{\mathcal{D}}_i^\alpha W = 0$, and the fact that \mathcal{L}^{++} , Σ^{++} and $\bar{\Sigma}^{++}$ obey the constraints (1.1). This result can be interpreted as a coupling of two vector multiplets described by the covariantly chiral field strengths W and \mathbb{W} , respectively.

$$\begin{aligned}
 S(\mathcal{L}^{++}) &= - \int d^4x d^4\theta \mathcal{E} W \mathbb{W}, \\
 \mathbb{W} &= -\frac{1}{8\pi} \oint \frac{(u^+ du^+)}{(u^+ u^-)^2} \left((\bar{\mathcal{D}}^-)^2 + 4\bar{S}^{--} \right) \mathbb{V}, \quad \mathbb{V} := \frac{\mathcal{L}^{++}}{\Sigma^{++}}. \tag{5.3}
 \end{aligned}$$

The composite superfield \mathbb{V} introduced can be interpreted as a tropical prepotential for the vector multiplet described by \mathbb{W} .

Let us choose the Lagrangian in (5.2) to be $\mathcal{L}^{++} = H^{++}\lambda$, where $H^{++}(z, u^+)$ is a tensor multiplet, and $\lambda(z, u^+)$ an arctic multiplet. Since both H^{++} and λ are independent of the vector multiplet described by the strengths W and \bar{W} , the latter can be chosen such that $\Sigma^{++} = H^{++}$. Then

$$S(H^{++}\lambda) = \frac{1}{8\pi} \int d^4x d^4\theta \mathcal{E} W \left((\bar{\mathcal{D}}^-)^2 + 4\bar{S}^{--} \right) \oint \frac{(u^+ du^+)}{(u^+ u^-)^2} \lambda(z, u^+). \tag{5.4}$$

We can now represent u^{+i} in the form (4.56) and fix the projective invariance by choosing u_i^- as in (4.57).

$$S(H^{++}\lambda) = -\frac{1}{8\pi} \int d^4x d^4\theta \mathcal{E} W \left((\bar{\mathcal{D}}^\perp)^2 + 4\bar{S}^{\perp\perp} \right) \oint d\zeta \lambda(z, \zeta) = 0, \tag{5.5}$$

since the integrand of the contour integral possesses no pole in the ζ -plane. This completes our second proof of the fact that the vector-tensor coupling $\mathcal{L}_{v-t}^{++} = H^{++}V$, with $H^{++}(z, u^+)$ is a tensor multiplet and $V(z, u^+)$ the tropical prepotential of a vector multiplet, is invariant under the gauge transformations (4.60).

In ref. [6], it was postulated that any chiral integral of the form

$$S_c = \int d^4x d^4\theta \mathcal{E} \mathcal{L}_c + \text{c.c.}, \quad \bar{\mathcal{D}}_\alpha \mathcal{L}_c = 0, \tag{5.6}$$

can be represented as follows:

$$S_c = \frac{1}{2\pi} \oint (u^+ du^+) \int d^4x d^4\theta d^4\bar{\theta} E \frac{W\bar{W}\mathcal{L}_c^{++}}{(\Sigma^{++})^2},$$

$$\mathcal{L}_c^{++} = -\frac{1}{4}V \left\{ \left((\mathcal{D}^+)^2 + 4S^{++} \right) \frac{\mathcal{L}_c}{W} + \left((\bar{\mathcal{D}}^+)^2 + 4\bar{S}^{++} \right) \frac{\bar{\mathcal{L}}_c}{\bar{W}} \right\}, \quad (5.7)$$

with $V(z, u^+)$ a tropical prepotential for the vector multiplet characterized by the field strength W . This assertion can now be immediately proved with the aid of (5.2).

Acknowledgments

We are grateful to Ian McArthur for reading the manuscript. This work was supported in part by the Australian Research Council. At a final stage of this project, G.T.-M. was supported by the endowment of the John S. Toll Professorship, the University of Maryland Center for String & Particle Theory, and National Science Foundation Grant PHY-0354401.

A Superspace geometry of conformal supergravity

Consider a curved 4D $\mathcal{N} = 2$ superspace $\mathcal{M}^{4|8}$ parametrized by local bosonic (x) and fermionic ($\theta, \bar{\theta}$) coordinates $z^M = (x^m, \theta_i^\mu, \bar{\theta}_{\dot{\mu}}^i)$, where $m = 0, 1, \dots, 3$, $\mu = 1, 2$, $\dot{\mu} = 1, 2$ and $i = \underline{1}, \underline{2}$. The Grassmann variables θ_i^μ and $\bar{\theta}_{\dot{\mu}}^i$ are related to each other by complex conjugation: $\bar{\theta}_{\dot{\mu}}^i = \bar{\theta}^{\dot{\mu}i}$. The structure group is chosen to be $\text{SO}(3,1) \times \text{SU}(2)$ [1, 54], and the covariant derivatives $\mathcal{D}_A = (\mathcal{D}_a, \mathcal{D}_\alpha^i, \bar{\mathcal{D}}_{\dot{\alpha}i})$ have the form

$$\mathcal{D}_A = E_A + \Omega_A + \Phi_A. \quad (A.1)$$

Here $E_A = E_A^M(z) \partial_M$ is the supervielbein, with $\partial_M = \partial/\partial z^M$,

$$\Omega_A = \frac{1}{2} \Omega_A^{bc} M_{bc} = \Omega_A^{\beta\gamma} M_{\beta\gamma} + \bar{\Omega}_A^{\dot{\beta}\dot{\gamma}} \bar{M}_{\dot{\beta}\dot{\gamma}} \quad (A.2)$$

is the Lorentz connection,

$$\Phi_A = \Phi_A^{kl} J_{kl}, \quad J_{kl} = J_{lk} \quad (A.3)$$

is the $\text{SU}(2)$ -connection. The Lorentz generators with vector indices ($M_{ab} = -M_{ba}$) and spinor indices ($M_{\alpha\beta} = M_{\beta\alpha}$ and $\bar{M}_{\dot{\alpha}\dot{\beta}} = \bar{M}_{\dot{\beta}\dot{\alpha}}$) are related to each other by the rule:

$$M_{ab} = (\sigma_{ab})^{\alpha\beta} M_{\alpha\beta} - (\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}} \bar{M}_{\dot{\alpha}\dot{\beta}}, \quad M_{\alpha\beta} = \frac{1}{2} (\sigma^{ab})_{\alpha\beta} M_{ab}, \quad \bar{M}_{\dot{\alpha}\dot{\beta}} = -\frac{1}{2} (\tilde{\sigma}^{ab})_{\dot{\alpha}\dot{\beta}} M_{ab}.$$

The generators of $\text{SO}(3,1) \times \text{SU}(2)$ act on the covariant derivatives as follows:¹⁸

$$\begin{aligned} [J_{kl}, \mathcal{D}_\alpha^i] &= -\delta_{(k}^i \mathcal{D}_{l)} \alpha, & [J_{kl}, \bar{\mathcal{D}}_{\dot{\alpha}i}] &= -\varepsilon_{i(k} \bar{\mathcal{D}}_{l)}^{\dot{\alpha}}, \\ [M_{\alpha\beta}, \mathcal{D}_\gamma^i] &= \varepsilon_{\gamma(\alpha} \mathcal{D}_{\beta)}^i, & [\bar{M}_{\dot{\alpha}\dot{\beta}}, \bar{\mathcal{D}}_{\dot{\gamma}i}] &= \varepsilon_{\dot{\gamma}(\dot{\alpha}} \bar{\mathcal{D}}_{\dot{\beta})}^i, & [M_{ab}, \mathcal{D}_c] &= 2\eta_{c[a} \mathcal{D}_{b]}, \end{aligned} \quad (A.4)$$

¹⁸In what follows, the (anti)symmetrization of n indices is defined to include a factor of $(n!)^{-1}$.

while $[M_{\alpha\beta}, \bar{\mathcal{D}}_{\dot{\gamma}}^i] = [\bar{M}_{\dot{\alpha}\dot{\beta}}, \mathcal{D}_{\dot{\gamma}}^i] = [J_{kl}, \mathcal{D}_a] = 0$. Our notation and conventions correspond to [1, 12]; they almost coincide with those used in [10] except for the normalization of the Lorentz generators, including a sign in the definition of the sigma-matrices σ_{ab} and $\tilde{\sigma}_{ab}$.

The supergravity gauge group is generated by local transformations of the form

$$\delta_K \mathcal{D}_A = [K, \mathcal{D}_A], \quad K = K^C(z) \mathcal{D}_C + \frac{1}{2} K^{cd}(z) M_{cd} + K^{kl}(z) J_{kl}, \quad (\text{A.5})$$

with the gauge parameters obeying natural reality conditions, but otherwise arbitrary. Given a tensor superfield $U(z)$, with its indices suppressed, it transforms as follows:

$$\delta_K U = K U. \quad (\text{A.6})$$

The covariant derivatives obey (anti-)commutation relations of the form:

$$[\mathcal{D}_A, \mathcal{D}_B] = T_{AB}{}^C \mathcal{D}_C + \frac{1}{2} R_{AB}{}^{cd} M_{cd} + R_{AB}{}^{kl} J_{kl}, \quad (\text{A.7})$$

where $T_{AB}{}^C$ is the torsion, and $R_{AB}{}^{kl}$ and $R_{AB}{}^{cd}$ constitute the curvature. The torsion is subject to the following constraints [54]:

$$\begin{aligned} T_{\alpha\beta}{}^{j\dot{c}} &= T_{\alpha\beta k}{}^{j\dot{\gamma}} = T_{\alpha\beta\dot{\gamma}}{}^{jk} = T_{\alpha\dot{j}k}{}^{i\dot{\beta}\gamma} = T_{a\beta}{}^{j\dot{c}} = T_{ab}{}^c = 0, \\ T_{\alpha\dot{j}}{}^{i\dot{\beta}c} &= -2i\delta_j^i (\sigma^c)_{\alpha}{}^{\dot{\beta}}, \quad T_{a\beta k}{}^{j\dot{\gamma}} = \delta_k^j T_{a\beta}{}^{\dot{\gamma}}. \end{aligned} \quad (\text{A.8})$$

Here we have omitted some constraints which follow by complex conjugation. The algebra of covariant derivatives is [1]

$$\begin{aligned} \{\mathcal{D}_{\alpha}^i, \mathcal{D}_{\beta}^j\} &= 4S^{ij} M_{\alpha\beta} + 2\varepsilon^{ij} \varepsilon_{\alpha\beta} Y^{\gamma\delta} M_{\gamma\delta} + 2\varepsilon^{ij} \varepsilon_{\alpha\beta} \bar{W}^{\dot{\gamma}\delta} \bar{M}_{\dot{\gamma}\delta} \\ &\quad + 2\varepsilon_{\alpha\beta} \varepsilon^{ij} S^{kl} J_{kl} + 4Y_{\alpha\beta} J^{ij}, \end{aligned} \quad (\text{A.9a})$$

$$\begin{aligned} \{\bar{\mathcal{D}}_i^{\dot{\alpha}}, \bar{\mathcal{D}}_j^{\dot{\beta}}\} &= -4\bar{S}_{ij} \bar{M}^{\dot{\alpha}\dot{\beta}} - 2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{Y}^{\gamma\delta} \bar{M}_{\dot{\gamma}\delta} - 2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} W^{\gamma\delta} M_{\gamma\delta} \\ &\quad - 2\varepsilon_{ij} \varepsilon^{\dot{\alpha}\dot{\beta}} \bar{S}^{kl} J_{kl} - 4\bar{Y}^{\dot{\alpha}\dot{\beta}} J_{ij}, \end{aligned} \quad (\text{A.9b})$$

$$\{\mathcal{D}_{\alpha}^i, \bar{\mathcal{D}}_j^{\dot{\beta}}\} = -2i\delta_j^i (\sigma^c)_{\alpha}{}^{\dot{\beta}} \mathcal{D}_c + 4\delta_j^i G^{\delta\dot{\beta}} M_{\alpha\delta} + 4\delta_j^i G_{\alpha\dot{\gamma}} \bar{M}^{\dot{\gamma}\dot{\beta}} + 8G_{\alpha}{}^{\dot{\beta}} J^i{}_j, \quad (\text{A.9c})$$

$$\begin{aligned} [\mathcal{D}_a, \mathcal{D}_{\beta}^j] &= i(\sigma_a)_{(\beta}{}^{\dot{\beta}} G_{\dot{\gamma})\dot{\beta}} \mathcal{D}^{\dot{\gamma}j} + \frac{i}{2} \left((\sigma_a)_{\beta\dot{\gamma}} S^{jk} - \varepsilon^{jk} (\sigma_a)_{\beta}{}^{\dot{\delta}} \bar{W}_{\dot{\delta}\dot{\gamma}} - \varepsilon^{jk} (\sigma_a)_{\dot{\gamma}}{}^{\alpha} Y_{\alpha\beta} \right) \bar{\mathcal{D}}_k^{\dot{\gamma}} \\ &\quad + \frac{i}{2} \left((\sigma_a)_{\beta}{}^{\dot{\delta}} T_{cd\dot{\delta}}{}^j + (\sigma_c)_{\beta}{}^{\dot{\delta}} T_{ad\dot{\delta}}{}^j - (\sigma_d)_{\beta}{}^{\dot{\delta}} T_{ac\dot{\delta}}{}^j \right) M^{cd} \\ &\quad + \frac{i}{2} \left((\tilde{\sigma}_a)^{\dot{\gamma}\gamma} \varepsilon^{j(k} \bar{\mathcal{D}}_{\dot{\gamma}}^{l)} Y_{\beta\gamma} - (\sigma_a)_{\beta\dot{\gamma}} \varepsilon^{j(k} \bar{\mathcal{D}}_{\dot{\delta}}^{l)} \bar{W}^{\dot{\gamma}\dot{\delta}} - \frac{1}{2} (\sigma_a)_{\beta}{}^{\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\gamma}}^j S^{kl} \right) J_{kl}, \end{aligned} \quad (\text{A.9d})$$

$$\begin{aligned} [\mathcal{D}_a, \bar{\mathcal{D}}_j^{\dot{\beta}}] &= -i(\sigma_a)_{\alpha}{}^{\dot{\beta}} G^{\alpha\dot{\gamma}} \bar{\mathcal{D}}_{\dot{\gamma}j} + \frac{i}{2} \left((\tilde{\sigma}_a)^{\dot{\beta}\gamma} \bar{S}_{jk} - \varepsilon_{jk} (\sigma_a)_{\alpha}{}^{\dot{\beta}} W^{\alpha\gamma} - \varepsilon_{jk} (\sigma_a)_{\dot{\alpha}}{}^{\gamma} \bar{Y}^{\dot{\alpha}\dot{\beta}} \right) \mathcal{D}_{\gamma}^k \\ &\quad + \frac{i}{2} \left((\tilde{\sigma}_a)_{\delta}{}^{\dot{\beta}} T_{cd\dot{\delta}}{}^{\delta} + (\sigma_c)_{\delta}{}^{\dot{\beta}} T_{ad\dot{\delta}}{}^{\delta} - (\sigma_d)_{\delta}{}^{\dot{\beta}} T_{ac\dot{\delta}}{}^{\delta} \right) M^{cd} \\ &\quad + \frac{i}{2} \left(-(\sigma_a)_{\dot{\gamma}}{}^{\gamma} \delta_j^{(k} \mathcal{D}_{\dot{\gamma}}^{l)} \bar{Y}^{\dot{\beta}\dot{\gamma}} - (\sigma_a)_{\dot{\gamma}}{}^{\beta} \delta_j^{(k} \mathcal{D}_{\dot{\delta}}^{l)} W^{\gamma\delta} + \frac{1}{2} (\sigma_a)_{\alpha}{}^{\dot{\beta}} \mathcal{D}_j^{\alpha} \bar{S}^{kl} \right) J_{kl}, \end{aligned} \quad (\text{A.9e})$$

where

$$T_{ab}{}^\gamma = -\frac{1}{4}(\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\mathcal{D}_k^\gamma\bar{Y}_{\dot{\alpha}\dot{\beta}} + \frac{1}{4}(\sigma_{ab})^{\alpha\beta}\mathcal{D}_k^\gamma W_{\alpha\beta} - \frac{1}{6}(\sigma_{ab})^{\gamma\delta}\mathcal{D}_\delta^l\bar{S}_{kl}, \quad (\text{A.10a})$$

$$T_{ab}{}^k = -\frac{1}{4}(\sigma_{ab})^{\alpha\beta}\bar{\mathcal{D}}_\gamma^k Y_{\alpha\beta} + \frac{1}{4}(\tilde{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}\bar{\mathcal{D}}_\gamma^k\bar{W}_{\dot{\alpha}\dot{\beta}} - \frac{1}{6}(\tilde{\sigma}_{ab})_{\gamma\delta}\bar{\mathcal{D}}_l^{\dot{\delta}}S^{kl}. \quad (\text{A.10b})$$

Here the real four-vector $G_{\alpha\dot{\alpha}}$, the complex symmetric tensors $S^{ij} = S^{ji}$, $W_{\alpha\beta} = W_{\beta\alpha}$, $Y_{\alpha\beta} = Y_{\beta\alpha}$ and their complex conjugates $\bar{S}_{ij} := \overline{S^{ij}}$, $\bar{W}_{\dot{\alpha}\dot{\beta}} := \overline{W_{\alpha\beta}}$, $\bar{Y}_{\dot{\alpha}\dot{\beta}} := \overline{Y_{\alpha\beta}}$ obey additional constraints implied by the Bianchi identities. They comprise the dimension 3/2 identities [1, 54]:

$$\mathcal{D}_\alpha^k S_{kl} + \mathcal{D}_l^\gamma Y_{\gamma\alpha} = 0, \quad \mathcal{D}_\alpha^i S^{jk} = \bar{\mathcal{D}}_{\dot{\alpha}}^{(i} S^{jk)} = 0, \quad \mathcal{D}_{(\alpha}^i Y_{\beta\gamma)} = 0, \quad \mathcal{D}_\alpha^i \bar{W}_{\dot{\beta}\dot{\gamma}} = 0, \quad (\text{A.11})$$

$$\bar{\mathcal{D}}_k^{\dot{\alpha}} \bar{S}^{kl} + \bar{\mathcal{D}}_{\dot{\gamma}}^l \bar{Y}^{\dot{\gamma}\dot{\alpha}} = 0, \quad \bar{\mathcal{D}}_{\dot{\alpha}}^i \bar{S}_{jk} = \mathcal{D}_{(i}^\alpha \bar{S}_{jk)} = 0, \quad \bar{\mathcal{D}}_i^{\dot{\alpha}} \bar{Y}^{\dot{\beta}\dot{\gamma}} = 0, \quad \bar{\mathcal{D}}_i^{\dot{\alpha}} W^{\beta\gamma} = 0, \quad (\text{A.12})$$

$$\mathcal{D}_\alpha^i G_{\beta\dot{\gamma}} = -\frac{1}{4}\bar{\mathcal{D}}_{\dot{\gamma}}^i Y_{\alpha\beta} + \frac{1}{12}\varepsilon_{\alpha\beta}\bar{\mathcal{D}}_{\dot{\gamma}l}S^{il} - \frac{1}{4}\varepsilon_{\alpha\beta}\bar{\mathcal{D}}^{\dot{\delta}i}\bar{W}_{\dot{\gamma}\dot{\delta}}, \quad (\text{A.13})$$

$$\bar{\mathcal{D}}_i^{\dot{\alpha}} G^{\gamma\dot{\beta}} = \frac{1}{4}\mathcal{D}_i^\gamma \bar{Y}^{\dot{\alpha}\dot{\beta}} - \frac{1}{12}\varepsilon^{\dot{\alpha}\dot{\beta}}\mathcal{D}^{\gamma l}\bar{S}_{il} + \frac{1}{4}\varepsilon^{\dot{\alpha}\dot{\beta}}\mathcal{D}_{\delta i}W^{\gamma\delta}. \quad (\text{A.14})$$

It should be remarked that the constraints (A.8) are invariant under super-Weyl transformations generated by a covariantly chiral superfield σ

$$\bar{\mathcal{D}}_i^{\dot{\alpha}}\sigma = 0. \quad (\text{A.15})$$

The reader is referred to [1, 9] for the transformation laws of the covariant derivatives and torsion superfields under the super-Weyl transformations.

B Projective supermultiplets

A projective supermultiplet of weight n , $Q^{(n)}(z, u^+)$, is a scalar superfield that lives on $\mathcal{M}^{4|8}$, is holomorphic with respect to the isotwistor variables u_i^+ on an open domain of $\mathbb{C}^2 \setminus \{0\}$. The variable u_i^+ are constant and invariant under the structure group action. The projective supermultiplet of weight n is characterized by the following conditions:

- (i) it obeys the covariant analyticity constraints (1.1);
- (ii) it is a homogeneous function of u^+ of degree n , that is,

$$Q^{(n)}(z, c u^+) = c^n Q^{(n)}(z, u^+), \quad c \in \mathbb{C}^*; \quad (\text{B.1})$$

- (iii) gauge transformations (A.5) act on $Q^{(n)}$ as follows:

$$\begin{aligned} \delta_K Q^{(n)} &= \left(K^C \mathcal{D}_C + K^{ij} J_{ij} \right) Q^{(n)}, \\ K^{ij} J_{ij} Q^{(n)} &= -\frac{1}{(u^+ u^-)} \left(K^{++} D^{(-1,1)} - n K^{+-} \right) Q^{(n)}, \quad K^{\pm\pm} = K^{ij} u_i^\pm u_j^\pm, \end{aligned} \quad (\text{B.2})$$

where

$$D^{(-1,1)} = u^{-i} \frac{\partial}{\partial u^{+i}}, \quad D^{(1,-1)} = u^{+i} \frac{\partial}{\partial u^{-i}}. \quad (\text{B.3})$$

The transformation law (B.2) involves an additional isotwistor, u_i^- , which is subject to the only condition $(u^+u^-) := u^{+i}u_i^- \neq 0$, and is otherwise completely arbitrary. By construction, $Q^{(n)}$ is independent of u^- , i.e. $\partial Q^{(n)}/\partial u^{-i} = 0$, and hence $D^{(1,-1)}Q^{(n)} = 0$. One can see that $\delta_K Q^{(n)}$ is also independent of the isotwistor u^- , $\partial(\delta_K Q^{(n)})/\partial u^{-i} = 0$, due to (B.1).

More generally, a weight- n isotwistor superfield $U^{(n)}(z, u^+)$ is defined to live on $\mathcal{M}^{4|8}$, be holomorphic with respect to the isotwistor variables u_i^+ on an open domain of $\mathbb{C}^2 \setminus \{0\}$, and satisfy the conditions (ii) and (iii).

The operators \mathcal{D}_α^+ and $\bar{\mathcal{D}}_{\dot{\alpha}}^+$ obey the anti-commutation relations:

$$\{\mathcal{D}_\alpha^+, \mathcal{D}_\beta^+\} = 4Y_{\alpha\beta} J^{++} + 4S^{++}M_{\alpha\beta}, \quad \{\mathcal{D}_\alpha^+, \bar{\mathcal{D}}_{\dot{\beta}}^+\} = 8G_{\alpha\dot{\beta}} J^{++}, \quad (\text{B.4})$$

where $J^{++} := u_i^+ u_j^+ J^{ij}$ and $S^{++} := u_i^+ u_j^+ S^{ij}$. It follows from (B.2)

$$J^{++} Q^{(n)} = 0, \quad J^{++} \propto D^{(1,-1)}, \quad (\text{B.5})$$

and hence the covariant analyticity constraints (1.1) are consistent.

We refer the reader to [1, 9] for a more complete analysis of projective supermultiplets including their super-Weyl transformation laws and the definition of the ‘‘smile’’ (or analyticity-preserving) conjugation.

C On the chiral projector

In this appendix we prove that the right hand side of (5.1)

1. is invariant under arbitrary projective transformations (4.17), (4.18) and (4.19); and
2. is covariantly chiral.

The expression (5.1) is manifestly invariant under arbitrary re-scalings of u^+ , eq. (4.17), as well as under the α -transformations (4.18). It remains to check invariance under infinitesimal β -transformations (4.18), with the parameter $\beta(t)$ constrained as in (4.19). Applying the β -transformation gives

$$\delta\left((\bar{\mathcal{D}}^-)^2 + 4\bar{S}^{--}\right)\left((\bar{\mathcal{D}}^+)^2 + 4\bar{S}^{++}\right)U^{(-2)} = 4\beta D^{(-1,1)}\bar{S}^{++}\left((\bar{\mathcal{D}}^+)^2 + 4\bar{S}^{++}\right)U^{(-2)}. \quad (\text{C.1})$$

Then, using (4.19) and the identity

$$\frac{d}{dt}V^{(+2)} = 2\frac{(\dot{u}^+u^-)}{(u^+u^-)}V^{(+2)} - \frac{(\dot{u}^+u^+)}{(u^+u^-)}D^{(-1,1)}V^{(+2)}, \quad (\text{C.2})$$

which holds for any isotwistor superfield $V^{(+2)}$ of weight +2, such as the superfield $\bar{S}^{++}\left((\bar{\mathcal{D}}^+)^2 + 4\bar{S}^{++}\right)U^{(-2)}$ appearing on the right of (C.1), it follows that

$$\beta\frac{(\dot{u}^+u^+)}{(u^+u^-)^2}D^{(-1,1)}V^{(+2)} = -\frac{d}{dt}\left(\frac{\beta}{(u^+u^-)}V^{(+2)}\right). \quad (\text{C.3})$$

This indeed demonstrates that the right hand side of (5.1) is projective invariant.

Now let us prove that the right hand side of (5.1) is covariantly chiral. First of all, consider a weight-zero isotwistor superfield $P(z, u^+)$ such that

$$\bar{\mathcal{D}}_{\dot{\alpha}}^+ P = 0. \quad (\text{C.4})$$

An example of such a superfield is $((\bar{\mathcal{D}}^+)^2 + 4\bar{S}^{++})U^{(-2)}$. Using the identities

$$J^{--}P = -(u^+u^-)D^{(-1,1)}P, \quad (\text{C.5})$$

$$\bar{\mathcal{D}}_{\dot{\alpha}}^-((\bar{\mathcal{D}}^-)^2 + 4\bar{S}^{--}) = -4\bar{S}^{--}\bar{\mathcal{D}}^{\dot{\beta}-}\bar{M}_{\dot{\alpha}\dot{\beta}} - 4\bar{Y}_{\dot{\alpha}\dot{\beta}}\bar{\mathcal{D}}^{\dot{\beta}-}J^{--} - \frac{4}{3}(\bar{\mathcal{D}}_{\dot{\alpha}q}\bar{S}^{q-})J^{--}, \quad (\text{C.6})$$

$$\begin{aligned} [\bar{\mathcal{D}}_{\dot{\alpha}}^+, (\bar{\mathcal{D}}^-)^2]P &= \left(4(\bar{\mathcal{D}}_{\dot{\alpha}}^-\bar{S}^{+-}) - 4(u^+u^-)\bar{Y}_{\dot{\alpha}\dot{\beta}}\bar{\mathcal{D}}^{\dot{\beta}-} \right. \\ &\quad \left. - 4D^{--}\bar{S}^{++}\bar{\mathcal{D}}_{\dot{\alpha}}^- - 2D^{--}(\bar{\mathcal{D}}_{\dot{\alpha}}^-\bar{S}^{++})\right)P, \end{aligned} \quad (\text{C.7})$$

one can show that

$$\begin{aligned} \bar{\mathcal{D}}_{\dot{\alpha}k} \left((\bar{\mathcal{D}}^-)^2 + 4\bar{S}^{--} \right) P &= \frac{1}{(u^+u^-)} \left[u_k^+ \bar{\mathcal{D}}_{\dot{\alpha}}^- - u_k^- \bar{\mathcal{D}}_{\dot{\alpha}}^+ \right] \left((\bar{\mathcal{D}}^-)^2 + 4\bar{S}^{--} \right) P \\ &= \frac{D^{(-1,1)}}{(u^+u^-)} \left(2u_k^- (\bar{\mathcal{D}}_{\dot{\alpha}}^-\bar{S}^{++}) + 4u_k^- \bar{S}^{++}\bar{\mathcal{D}}_{\dot{\alpha}}^- + 4u_k^+ (u^+u^-)\bar{Y}_{\dot{\alpha}\dot{\beta}}\bar{\mathcal{D}}^{\dot{\beta}-} - 2u_k^+ (\bar{\mathcal{D}}_{\dot{\alpha}}^+\bar{S}^{--}) \right) P. \end{aligned} \quad (\text{C.8})$$

Secondly, we notice that for any superfield $V_k^{(+2)}(z, u^+)$, which is homogeneous in u_i^+ of degree +2, the following identity holds

$$\frac{(\dot{u}^+u^+)}{(u^+u^-)^3} D^{(-1,1)} V_k^{(+2)} = -\frac{d}{dt} \left(\frac{1}{(u^+u^-)^2} V_k^{(+2)} \right). \quad (\text{C.9})$$

Using (C.8) and (C.9), one can then prove that

$$\bar{\mathcal{D}}_{\dot{\alpha}k} \oint \frac{(u^+du^+)}{(u^+u^-)^2} \left((\bar{\mathcal{D}}^-)^2 + 4\bar{S}^{--} \right) \left((\bar{\mathcal{D}}^+)^2 + 4\bar{S}^{++} \right) U^{(-2)} = 0. \quad (\text{C.10})$$

As a result, the right hand side of (5.1) is indeed covariantly chiral.

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